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A

PRACTICAL TREATISE

ON

ALGEBRA,

DESIGNED FOR THE USE OF STUDENTS

IN

HIGH SCHOOLS AND ACADEMIES.

By BENJAMIN GREENLEAF, A. M.

AUTHOR OF THE "NATIONAL ARITHMETIC," ETC.

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PREFACE.

THE following Treatise is designed to present a system of theoretical and practical Algebra. It is intended to be both elementary and comprehensive, and adapted to the wants of beginners, as well as those who are advanced in the study.

In the course of his labors the author has consulted the most approved European treatises on the subject, and availed himself of whatever he thought might add to the interest and usefulness of his work.

It has been the aim of the author throughout his investigations, to give to it a practical character, that those who study it may know how to apply their knowledge to useful purposes.

It will be seen, that a place is given in this volume to the subject of Proportion, which was deemed desirable on account of those, who may study Algebra before they enter upon Geometry.

The demonstrations connected with the several Roots, will greatly aid those who wish for a complete and thorough knowledge of Evolution in Arithmetic.

A large number of Examples have been placed under each Rule; and the pupil, who will carefully and intelligently perform them, will gain a more thorough knowledge of the principles of the science, than he could do by any other means.

Long experience in teaching has convinced the author, that in both Arithmetic and Algebra, it is highly important for the learner to apply his knowledge to practice; and, to this end, he should solve a variety of problems, and some of them difficult; in this way, and in no other, as the author conceives, can one become a thorough and accomplished mathematician.

The subject of minus and plus on page 29, and also the completing the square in Affected Equations, page 166, the author has illustrated by diagrams. This method he has seen in no other work.

The method of solving Cubic Equations by completing the square, on page 181, he believes will be very useful. This method will not apply to all problems, but wherever it will apply, it not only very much abridges the labor, but the result is perfect accuracy, which is not always the fact by the common method of approximation. The Table of Logarithms at the end of the work, will be often found convenient and useful.

The examples presented in this work are intended to be neither too numerous nor too difficult; and all who may use it, either by themselves, or in connection with a class, are advised to solve all the problems in the order in which they are given. No labor on the part of the pupil will be productive of more intellectual and practical benefit. The answers to several questions have been designedly omitted, that the pupil may try his skill as upon an original problem.

One, who has a thorough knowledge of Arithmetic, will find the study of Algebra a most pleasing, and, generally, not a difficult task. It will lead him to a knowledge of many facts and principles, which will be likely to surprise him with delight. As a mental exercise, it is exceedingly pleasant and profitable, and is admirable for its effect upon the logical powers of the mind, assisting one to think and reason closely and conclusively.

It is a remark of Mr. Locke in his Essay on the Human Understanding, that "nothing teaches a man to reason so well as Mathematics, which should be taught to all those who have time and opportunity, not so much to make them mathematicians, as to make them reasonable creatures." This may well be said of Algebra.

In the study of Algebra, and in fact of every branch of mathematics, the author would recommend, that a good portion of time, say from one fifth to one third, be devoted to reviewing. The pupil may be assured that no hours will be better spent. Nothing will do more to make him master of any subject; nothing so much to fix the important principles in his mind, and show him their numerous applications and their wide-spread relations.

BENJAMIN GREENLEAF.

BRADFORD, *January* 23, 1852.

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ERRATA.

In printing the first edition of a work of this kind, it is impracticable to avoid errors; the most important ones are here noticed.

Page 10, line 6, after the word figures, insert "or some letter or letters."
" 11, line 22, for $5a$ read $a=5$.
" 16, line 1, for $\surd 2a+c^2$, read $\sqrt{2ac+c^2}$.
" 18, line 7, for $7ax$ read $-7ax$.
" 18, line 19, for "are unlike" read "are alike."
" 24, line 18, for $10x$ read $-10x$.
" 25, line 12, for $7abc$ read $7abc^2$.
" 33, line 19, for $4x^2y$ read $3x^2y$.
" 34, line 24, for a^4-x^2 read a^4-x^4.
" 35, line 2, for $5x^4$ read $5a^4$.
" 36, line 27, for $+7b$ read $-7b$.
" 41, lines 7, 8, 9, the signs of a^2x should be changed.
" 43, lines 11 and 12, for c^2 and y^5 read b^2 and y^4.
" 48, line 10, for $a+x$ read a^2-x^2.
" 49, line 5, for nq read q.
" 49, line 11, for $\frac{bd^2}{4a^2e^2}$ read $\frac{b}{4a^2e^2}$.
" 51, line, 6, a^2 should be added to the numerator.
" 53, line 11, for $6\frac{3}{5}$ read $6\frac{3}{4}$.
" 60, lines 16 and 25, for $\frac{55}{80}$ and $a-1$ read $\frac{45}{80}$ and a^2-1.
" 65, line 3, after Negative Exponents, erase a.
" 65, line 5, for $10b^4$ read $10ab^8$.
" 67, line 2, erase a.
" 67, line 4, read $\frac{a^2x^3y^2-x^3y^2-a^2x^3y^4}{a^2x^3y^2-a^4+a^2xy^2}$.
" 81, line 27, for $\frac{4}{5}$, read $\frac{2}{5}$.
" 113, line 18, the word equal should be omitted.
" 116, line 7, for square $2a+b$ read square $a+b$.
" 123, lines 9 and 11 for y^3 read y^2.
" 126, line 19, for $x^2-x+\frac{1}{2}$, read $a^2-a+\frac{1}{2}$.
" 130, line 24, for 90 read 96.
" 142, line 29, for $14a$ read $14a^2$.
" 159, line 3, for $\sqrt{n}$ read $x\sqrt{n}$.
" 167, line 10, for $-5+7$ read $-5-7$.
" 168, lines 17, 18, and 19, for $-\frac{1}{36}, -\frac{1}{6}, \frac{1}{3}$ read $\frac{1}{36}, \frac{1}{6}, -\frac{1}{3}$.
" 178, lines 13 and 14, for 746 and 253 read 756 and 243.
" 181, line 24, for "in its surface" read "in half its surface."
" 182, line 19, omit "sum he gave square."
" 187, lines 20 and 23, for 2 to 9, and 4 to 27, read 9 to 2, and 9 to 16.
" 191, lines 1 and 3, for 50 and 75 read 75 and 50.
" 197, line 29, for 12 : 6 read 6 : 12.
" 202, line 9, for $a+bc$ read $a+b$.
" 223, line 3, for 3 read 2
" 233, line 30, for $\frac{29}{125}$ read $\frac{77}{125}$.
" 234, line 16, for 10 read 9.
" 235, line 11, for 64 read 48.
" 241, line 29, for \$3.99 read \$1.01.
" 246, line 30, for 1727 read 1827.
" 247, line 12, for n read x.
" 281, line 18, for 0.051189 read 0.021189.
" 315, line 5, after obliged, insert, "the last two days."

ALGEBRA.

SECTION I.

DEFINITIONS AND NOTATIONS.

ARTICLE **1.** ALGEBRA is the art of computing by symbols.

2. In Arithmetic, we represent quantities, and perform calculations by the numerical figures, each of which has a known and definite value.

3. In Algebra, we employ the letters of the alphabet, and other characters, the value of which is either known or unknown, according to the conditions of the problems.

4. Those quantities whose values are given, are called *known* quantities; and those whose value is not given, *unknown* quantities.

5. The symbols, used to denote known quantities, are, generally, the first letters of the alphabet in the small or Italic character, as a, b, c, d, &c.; and those used to denote unknown quantities, the last letters, as w, x, y, z.

6. In addition to the above, which are the more common symbols, capital letters may be used, as A, B, C, D, &c., or letters of the Greek alphabet, as β, γ, δ, ϵ, ζ, &c. In extensive operations, the use of these or some other suitable characters is sometimes very convenient.

7. Sometimes quantities are expressed in Algebra as in Arithmetic, by figures, instead of letters.

8. When a quantity is doubled, or trebled, or multiplied any number of times, the number of multiplications is usually expressed by a numerical figure or figures. Thus, let a denote a certain quantity, and $2a$ will denote twice the same quantity, $3a$ three times the same quantity, &c.

9. The numerical figure or figures prefixed to any symbol, and denoting the number of times the quantity, represented by the symbol, is taken, is called the *coefficient*. Thus, in $4a$, $7b$, and $9c$, the coefficients are 4, 7, and 9.

10. A quantity which has no numerical figure prefixed to it, is considered as having a unit for its coefficient. Thus, a is the same as $1a$.

11. Quantities represented by the same symbol, or letters, and of the same power, are called like quantities; and those represented by different symbols or letters, *unlike* quantities. Thus, $3a$, $4a$, and $5a$, are like quantities, and $3a$, $4b$, and $5c$, unlike quantities. In like manner, $3ab$, $4ab$, and $6ab$ are like quantities, and $3ab$, $4ac$, $5dc$, and $6mn$ are unlike quantities.

12. Besides the symbols and figures used to denote quantity, there are certain signs, which are used to express the different relations between quantities, and the operations to which these quantities are subjected. These signs are the same as are often employed in Arithmetic; but, in Algebra, they are indispensable.

13. The sign $=$ is that of *equality*, and denotes that the two quantities between which it is placed are equal to each other. Thus, $a=2b$, signifies that a is equal to $2b$.

14. The sign $+$ is called *plus*, and signifies addition. Thus, $a+b$ signifies that a is to be added to b.

15. The sign $-$ is called *minus*, and signifies subtraction. Thus, $a-b$ signifies that b is to be substracted from a.

16. Sometimes both the signs $+$ and $-$ occur before the same quantity, as $a\pm x$, in which case they signify that the quantity may be either added or subtracted, or that it is doubtful which operation is to be performed.

17. The sign $\times$ signifies multiplication. Thus, $a \times b$ denotes that a is to be multiplied by b; and $a \times b \times c \times d$, that the quantities a, b, c, and d are to be multiplied together. This sign is read *into*. Thus, $a \times b$ is to be read, a into b. Sometimes a single point is substituted for $\times$. Thus, $a.b$ signifies that a is multiplied by b.

18. The sign $\div$ signifies division. Thus, $a \div b$ signifies that a is to be divided by b.

19. Division is also represented by placing the divisor under the dividend, in the form of a fraction. Thus, $\frac{a}{b}$ signifies that a is to be divided by b; and $\frac{a-b}{a+b}$ that $a-b$ is to be divided by $a+b$.

20. The sign $>$, standing between two quantities, denotes that the one before it is greater than the one after it. Thus, $a > b$ signifies that the quantity a is greater than the quantity b.

21. On the other hand, the sign $<$ denotes that the quantity before it is less than the one after it. Thus, $b < a$ signifies that b is less than a.

22. The sign $\therefore$ signifies therefore. Thus, $5a = \therefore 3a = 15$, is read, a is equal 5, therefore $3a$ is equal to 15.

23. The signs : :: : denote proportion. Thus, $a : b :: c : d$ is to be read, as a is to b, so is c to d; and the signs, placed in their order, indicate that a has the same ratio to b that c has to d.

24. The sign, $\surd$, called the radical sign, signifies the square root of the quantity, which follows it; or, that the root of the quantity is to be extracted. Thus, $\sqrt{a}$ denotes that the square root of a is to be extracted.

25. By placing a figure above the sign, thus, $\sqrt[3]{}$, it is made the radical sign of any root whatever. Thus, $\sqrt[3]{a}$ signifies the cube root of a; $\sqrt[4]{a}$, the fourth root of a; $\sqrt[5]{a}$, the fifth root; $\sqrt[m]{a}$, the *m*th root, &c.

26. The power of a quantity is denoted by a figure placed above it at the right. Thus, a^2 signifies the second power of a; a^3 the third power of a; a^4 the fourth power, &c.

27. In operating with unknown quantities, it is frequently necessary to express the root of a certain power of a quantity; as, for instance, the 4th root of the 3d power of a. In this case, a fraction is to be used; the numerator denoting the power to which the quantity is to be raised, and the denominator the root of the power. Thus, $a^{\frac{2}{3}}$ denotes the cube root of the second power of a; $b^{\frac{6}{4}}$, the fourth root of the sixth power of b. By inverting the fraction, and writing it before the radical sign, we may represent the same. Thus, $\sqrt[\frac{3}{2}]{a}$, $\sqrt[\frac{4}{6}]{b}, = a^{\frac{2}{3}}, b^{\frac{6}{4}}$.

28. When a quantity is represented by a single letter, or several letters, placed one after the other without the sign $+$ or $-$ between them, it is called a ***simple quantity.*** Thus, a, bc, cde, $3ab$ are simple quantities.

29. When a number of simple quantities are connected by the signs $+$ or $-$, the result is a *compound quantity*. Thus, $a+b$, $be+cd$, $4a+5cd-x$, are compound quantities.

30. A *term* is any part or number of a compound quantity, separated from the other parts by the sign $+$ or $-$. Thus, in the compound quantity $a+b$, a and b are the terms. So in $xy-y+z$, xy, and y, and z are the terms.

31. When two or more members of a compound quantity are to be subjected to the same operation, in which they are to be regarded as one whole, they are connected by a line drawn over them, called a *vinculum;* or by inclosing them in a parenthesis. Thus, when we are to multiply $a+b+c$ by any number, as 3, we write $\overline{a+b+c}\times 3$, or $(a+b+c)\times 3$, or more simply $3(a+b+c)$. So $\overline{x+y}\times\overline{y+z}$, or $(x+y)\,(y+z)$ signifies, that $x+y$ is to be regarded as a whole, and multiplied by $y+z$ taken also as a whole; whereas, if the line or parenthesis were not employed, $a+b\times 3$ would denote, that b only is to be multiplied by 3, and the result would be $a+3b$.

32. When two or more quantities are multiplied together, each quantity is called a *factor*. Thus in ab, a and b are called factors; so in cde, c, d, and e are severally called factors.

33. A *composite* number is one, which is produced by the multiplication of two or more quantities or factors into each other. Thus, the quantity abc is a composite one, the factors of which are a, b, and c.

34. Quantities, which have the sign plus before them, either expressed or implied, are called *positive* or *affirmative* quantities. Thus $+a$, $+b$, or a, b, are positive or affirmative, the sign $+$ being always implied before a quantity, which has no express sign prefixed.

35. Quantities, which have the sign — prefixed, are called *negative* quantities. Thus $-a$, $-b$, -3, are negative quantities.

36. Where the signs are all positive or all negative, they are called *like* signs.

37. When some of the signs are positive and others negative, they are said to be *unlike*.

38. When a quantity consists of one term, it is called a *monomial*, as a, ab, $3xy$, being the same as a simple quantity.

39. When a quantity consists of two terms, it is called a *binomial*. Thus $a+b$, $x+y$, are called binomial quantities; and $a-b$ a residual binomial.

40. When a quantity consists of three terms, it is called a *trinomial*. Thus, $a+b+c$, or $x+y+z$, is a trinomial.

41. When a quantity consists of any number of terms greater than three, it is called a *polynomial*. Thus $a+b+c+d$, or $w+x+y+z$, are polynomials.

42. The *power* of a quantity is its square, cube, biquadrate, &c.; called also its second, third, fourth power, &c.; as a^2, a^3, a^4, &c.

43. The *index* or *exponent* of a quantity, is the number which denotes its power or root.

Thus, -1 is the index or exponent of a^{-1}; 2 is the index of a^2; $\frac{1}{2}$, of $a^{\frac{1}{2}}$, or $\sqrt{a}$; and m and $\frac{1}{n}$, of a^m and $a^{\frac{1}{n}}$.

44. When a quantity appears without any index or exponent, it is always understood to have for it unity, or 1.

Thus, a is the same as a^1, $2x$ is the same as $2x^1$; the 1 in such cases being usually omitted.

45. A *rational quantity* is that, which can be expressed in finite terms, or without any radical sign or fractional index; as a, or $\frac{2a}{3}$, or $5a^2$, &c.

46. An *irrational quantity*, or sum, is that which has no exact root, or which can only be expressed by means of the radical sign, or a fractional index; as $\sqrt{a}$, or $2^{\frac{1}{2}}$, $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, &c.

47. A square or cube number, &c. is that, which has an exact square or cube root, &c.

Thus, 4 and $\frac{9}{16}a^2$ are square numbers; and 64 and $\frac{8}{27}a^3$, are cube numbers, &c.

48. A measure or divisor of any quantity, is that, which is contained in it some exact number of times, without a remainder.

Thus, 3 is the measure or divisor of 6, $7a$ is a measure of $35a$, and $9ab$ of $27a^2b^2$.

49. *Commensurable numbers* or quantities, are such as have a common measure or divisor, or that can be each divided by the same quantity without a remainder.

Thus, 6 and 8, $2\sqrt{2}$ and $3\sqrt{2}$, $5a^2b$ and $7a^2b$, are commensurable quantities; the common divisors being 2, $\sqrt{2}$, and *ab*.

50. A *prime number*, is that which has no exact divisor, except itself or unity; as, 2, 5, 7, 11, 13, 17, &c.; and the intervening numbers, 4, 6, 8, 9, 10, 12, 14 and 16, are *composite numbers.*

51. Two or more numbers are said to be *incommensurable*, or prime to each other, when they have no common measure or divisor, except unity; as 2 and 3, 5 and 7, 17 and 19, &c.

52. One quantity is called the *multiple* of another, when the former contains the latter a certain number of times without a remainder.

Thus, $15a$ is a multiple of $5a$, and $6a$ of $3a$.

53. The *reciprocal* of any quantity, is that quantity inverted, or unity divided by it.

Thus, the reciprocal of a or $\frac{a}{1}$ is $\frac{1}{a}$; the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and the reciprocal of $\frac{a+b}{a-b}$ is $\frac{a-b}{a+b}$.

54. A *series* is a rank or succession of quantities, which usually proceed according to some certain law; as $1+\frac{1}{2}a+\frac{1}{4}a^2+\frac{1}{8}a^3+\frac{1}{16}a^4$, &c.

PRACTICAL EXAMPLES.

55. In calculating numerical values of Algebraic Expressions, let $a=6$, $b=5$, $c=4$, $d=1$, and $e=0$.

1. $a^2+2ab-c+d=36+60-4+1=93$.
2. $2a^3-3a^2b+c^3=432-540+64=-44$.
3. $a^2\times(a+b)-2abc=36\times11-240=156$.

4. $2a\sqrt{b^2-ac}+\sqrt{2a+c^2}=12\times1+8=20.$

5. $3a\sqrt{2ac+c^2}$, or $3a(2ac+c^2)^{\frac{1}{2}}=18\sqrt{64}=144.$

6. $\sqrt{(2a^2-\sqrt{6ac+e^2})}=\sqrt{(72-\sqrt{144})}=\sqrt{60}.$

7. $\sqrt{(a^2b+4b^2-5\sqrt{cd})}=\sqrt{(180+100-10)}=\sqrt{270}.$

8. $\sqrt{(ab^2+2b^3-5\sqrt{dc}}=\sqrt{(150+250-10)}=\sqrt{390}.$

9. $\frac{2a+3c}{6d+4e}+\frac{4bc}{\sqrt{2ac+c^2}}=\frac{24}{6}+\frac{80}{\sqrt{64}}=\frac{24}{6}+\frac{80}{8}=4+10=14.$

10. $\frac{2ab-5c}{4b-10d}+\left(\frac{3a-\frac{1}{2}bc}{5a-12d}\right)^{\frac{1}{2}}=\frac{40}{10}+\sqrt{\frac{18-10}{30-12}}=\frac{40}{10}+\sqrt{\frac{4}{9}}=$ $4\frac{2}{3}.$

11. $2a^2+3bc-5=$

12. $5a^2b-10ab^2+27e=$

13. $7a^2+(b-c)\times(d-e)=$

14. $\frac{ab^2}{c}\times d-\frac{a-b}{d}+27a^2e=$

15. $3\sqrt{c}+2a\sqrt{(2a+b-d)}=$

16. $a\sqrt{(a^2+b^2)}-3bc\sqrt{(a^2-b^2)}=$

17. $3a^2b+\sqrt[3]{(c^2-\sqrt{2ac+c^2})}-3e=$

18. $\frac{2b-c}{3a-c}-\frac{\sqrt{5b-3\sqrt{c}+d}}{2a+c}+3a^3\sqrt{(a+b)}=$

SECTION II.

ADDITION.

Art. 56. Addition in Algebra is the connecting together of several quantities by their appropriate signs, so that the result shall express their proper sum.

57. The operation consists in collecting into one term all the similar and like quantities, and so arranging the several terms, thus obtained, as by signs, to indicate the proper sum of all the quantities, both like and unlike.

58. Addition in Algebra properly embraces three cases.

I. When the quantities are alike, and their signs alike also; as, a, $3a$; or, $-b$, $-4b$.

II. When the quantities are alike, and their signs unlike; as, $3b$, $-5b$.

III. When the quantities are unlike, some having like and others unlike signs; as, $3a$, $4b$, $-4x$.

CASE I.

59. When the quantities are alike, and their signs alike.

RULE. *Add together the coefficients belonging to the like or common quantities, and place their sums before the common letter or letters, with the proper signs prefixed; and the result will be the sum required.*

Thus, let it be required to add together $3ab$, $7ab$, $8ab$, the operation will be as follows:—

$$\begin{array}{r} 3ab \\ 7ab \\ 8ab \\ \hline 18ab. \end{array}$$

The reason of this rule is obvious; for, since ab, whatever be its value, must represent the same quantity in every instance, it is evident that 3 times, 7 times, and 8 times the same quantity will make 18 times the same.

In like manner, let it be required to add together $-7b$, $-5b$, and $-6b$.

$$\begin{array}{r} -7b \\ -5b \\ -6b \\ \hline -18b. \end{array}$$

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$3a$	$7h$	$-3ax$	$4xy^2$	$3a-2y^2$
$4a$	$5h$	$-4ax$	xy^2	$4a-3y^2$
$6a$	$3h$	$-ax$	$3xy^2$	$6a-y^2$
a	$8h$	$-2ax$	$7xy^2$	$a-6y^2$
$5a$	h	$7ax$	$2xy^2$	$5a-2y^2$
$19a$	$24h$	$-17ax$	$17xy^2$	$19a-14y^2$

(6)	(7)	(8)	(9)	(10)
$7x$	$14abc$	$5y$	$-4mn$	$5h+x$
$4x$	$11abc$	y	$-3mn$	$h+2x$
$11x$	$5abc$	y	$-mn$	$2h+4x$
$9x$	$4abc$	$3y$	$11mn$	$h+x$
x	abc	$4y$	$-mn$	$7h+6x$

11. Add $7a$, $11a$, a, $4a$, $6a$, and $3a$ together. *Ans.*

12. Add $4h$, $6h$, h, h, $11h$ and $7h$ together. *Ans.*

13. Add together $(3a^2-b)$, $(7a^2-4b)$, and (a^2-b). *Ans.*

CASE II.

60. When the quantities are unlike, and have unlike signs.

RULE. *Add all the affirmative coefficients into one sum, and those that are negative into another; then subtract the least of these results from the greater, and prefix the sign of the greater to the difference, annexing the common letter or letters.*

Required the sum of $+7ax-4ax-3ax+17ax-ax$, and $+ax$.

$$\begin{array}{r} 7ax\\ -\ 4ax\\ -\ 3ax\\ 17ax\\ -\ ax\\ ax\\ \hline 17ax. \end{array}$$

We find the sum of the plus quantities to be $25ax$, and the sum of the negative quantities $-8ax$; and the difference between these coefficients is 17, which we apply to ax; thus, $17ax$.

The reason of this Rule is obvious, when we consider, that two equal quantities, the one with a positive and the other with a negative sign, exactly cancel each other, so that their sum is nothing. Of course, then, when two like quantities of opposite signs are not equal, the difference between them must be the proper sum, which will be positive or negative according to the affirmative or negative character of the larger quantity.

EXAMPLES.

(1)	(2)	(3)	(4)	(5)
$7a$	$-6m$	$4ax$	$13n$	$7a-mp^2$
$-3a$	m	$-3ax$	n	$a+6mp^2$
a	$-11m$	ax	$-20n$	$-11a-3mp^2$
$-5a$	$5m$	$-7ax$	$6n$	$8a+11mp^2$
$11a$	$-m$	ax	$8n$	$-9a-7mp^2$
a	$20m$	$12ax$	$-n$	$18a-15mp^2$
$12a$				

(6)	(7)	(8)	(9)	(10)
$6y$	$4mn$	$-8xy$	$-4pn$	$8xy-m^2p$
$-7y$	mn	$7xy$	pn	$3xy+m^2p$
$4y$	$3mn$	$-4xy$	pn	$-11xy-18m^2p$
$-11y$	$18mn$	$-xy$	$-11pn$	$-4xy+9m^2p$
$9y$	$7mn$	$9xy$	$7pn$	$-8xy-3m^2p$
$-2y$	$8mn$	$-3xy$	pn	$12x+12m^2p$

11. Add $4+a^2x$, $6-a^2x$, $3+6a^2x$, $15-5a^2x$, $3+a^2x$, and $6+7a^2x$ together. *Ans.* $37+9a^2x$.

12. Add $14ax-6y$, $7ax+y$, $5ax-7y$, $9xa-11y$, and $8ax+3y$ together. *Ans.* $43ax-20y$.

13. Add $3a-4b+6c$, $7a+11b-3c$, $8a+b-7c$, and $a-11b+15c$ together. *Ans.* $19a-3b+11c$.

14. Add $16x^2-5y^3-16$, $3x^2+4y^3-5$, x^2+3y^3-37, x^2-y^3+7, $6x^2+7y^3-11$, and $2x^2-3y^3-21$ together.
Ans. $29x^2+5y^3-83$.

15. Add $5a-b$, $3b+3c$, $4a-5c$, $5a-5b-c$, $7a-6c$, and $11a+4b-7c$ together. *Ans.* $32a+b-16c$.

Case III.

61. When the quantities are unlike, and some having like and others unlike signs. It is evident, that unlike quantities cannot be united into one ; or otherwise added, than by means of their signs.

Thus, for example, if a be supposed to represent 20, and b 12, then the sum of a and b can be neither twice 20 nor twice 12, but it must be $20+12=32$, that is, $a+b$.

62. Hence, to add unlike quantities, we have the following

Rule. *Collect all the like quantities together, as in* Case II, *and write down those that are unlike, one after another, with their proper signs.*

63. When several quantities are to be added together, it is immaterial in what order they are written.

Thus, $a+b-c$, $a-c+b$, $-c+a+b$, are equivalent expressions.

EXAMPLES.

(1)	(2)
$3ax$	$7\,a+7m$
$4mn$	$6\,a-7x$
$-6y^2$	$4xy-5m$
$7xy$	$3x\;+8xy$
$3ax+4mn-6y^2+7xy.$	$13a\;+12xy+2m-4x.$

(3)	(4)	(5)
$9x^2y$	$14ax - 2x^2$	$4ax-130+3x^{\frac{1}{2}}$
$-7x^2y$	$5ax^2+3xy$	$5a^2+3ax+9x^2$
$3axy$	$8y^2 - 4ax$	$7xy-4x^{\frac{1}{2}}+9^0$
$-4xy^2$	$3x^2+26$	$\sqrt{x}-40-6x^2$

6. Add together $a+b$ and $a-b$. *Ans.*

7. Add $3a+b-10$, $c-d-a$, $-4c+2a-3b-7$, and $4x^2+5-18m$ together. *Ans.*

8. Add $7a-5y^3$, $8\sqrt{x}+2a$, $5y^3-\sqrt{x}$, and $-9a+7\sqrt{x}$ together. *Ans.*

9. Add $4mn+3ab-4c$, $3x-4ab+2mn$, and $3m^2-4p$ together. *Ans.*

10. Find the sum of $3a^2+2ab+4b^2$, $5a^2-8ab+b^2$, $-a^2+5ab-b^2$, $18a^2-20ab-19b^2$, and $14a^2-3ab+20b^2$. *Ans.*

11. Find the sum of $4x^3-5a^3-5ax^2+6a^2x$, $6a^3+3x^3+4ax^2+2a^2x$, $-17x^3+19ax^2-15a^2x$, $13ax^2-27a^2x+18a^3$, $3a^2x-20a^3+12x^3$, and $31a^2x-2x^3-31ax^2-7x^3$. *Ans.*

64. Coefficients, whether numeral figures or letters, that are common to several terms, may be connected with them by a parenthesis.

	(12)	(13)
Add	$amx+2dy$	$hy+4mx$
	$2cx-3dy$	$my+3dx$
	$3ax+5y$	$4ny-9mx$
	$(am+2c+3a)x+(5-d)y.$	$(h+m+4n)y+(3d-5m)x.$

14. Add $4ax-5my$, $3dx+7ny$, and $7mx+4my$ together. *Ans.* $(4a+3d+7m)x+(7n-m)y.$

15. Add $3hz-5x$, $4mz+nx$, and $5az-4px$ together. *Ans.* $(3h+4m+5a)z+(n-5-4p)x.$

SECTION III.

SUBTRACTION.

Art. 65. Subtraction is the taking of one quantity from another, or the method of finding the difference between any two quantities or sets of quantities of the same kind.

66. If it be required to subtract $10-7$ from 12, we might first subtract 7 from $10=3$, and take the three from $12=9$; or, we might take the 10 from 12, and the remainder 2 must necessarily be increased by 7 to produce the correct result.

If from a we wish to subtract $c-d$, we first subtract c, and it gives $a-c$. This quantity, since we have taken d too much from a, is too small by d. Therefore, d must be added, thus, $a-c+d$.

67. If a simple quantity is to be taken from another simple quantity, it is only necessary to write them one after the other; thus, if 8 is to be taken from 15, it may be expressed thus, $15-8=7$.

If it were required to subtract b from a, it should be written thus, $a-b$; but, if we were to subtract $a-b$ from $c+d$ it is evident that if only a were to be taken, it would be written thus, $c+d-a$. But this evidently gives a result too small; for a was to be lessened by b, before the subtraction. Therefore, as the remainder is too small by b, we must add b to the remainder, which will give $c+d-a+b$; for it makes no difference in the result, whether the minuend be increased or the subtrahend lessened.

Subtract $7-4$ from 13. Taking 7 from 13 leaves 6; but 6 is too small, for the 7 should have been lessened by 4, and we must either subtract the 4 from the 7 before the operation, or add it to the remainder; and, if added to the remainder, the expression will be thus, $6+4=10$.

68. We therefore see the propriety of the following

RULE. *Change the signs of all the quantities to be subtracted, and proceed as in addition; that is, the minus quantities in the subtrahend must be changed to plus quantities, and the plus quantities to minus quantities.*

SIMPLE QUANTITIES.

	(1)	(2)	(3)	(4)	(5)	(6)
From	$+7a$	$-16x$	$+17d$	$-29g$	$+15b$	$-6c$
Take	$+2a$	$-5x$	$+8d$	$-18g$	$+7b$	$-c$
	$+5a$	$-11x$	$+9d$	$-11g$	$+8b$	$-5c$

The above questions are performed as in Arithmetic, the minuend being the larger number, and having the same sign as the subtrahend.

	(7)	(8)	(9)	(10)	(11)	(12)	(13)
From	$-8a$	$+7x$	$+18y$	$-3b$	$-7c$	$+8d$	$-6h$
Take	$-15a$	$+14x$	$+20y$	$-7b$	$-15c$	$+11d$	$-8h$
	$+7a$	$-7x$	$-2y$	$+4b$	$+8c$	$-3d$	$+2h$

In these examples, the minuend is taken from the subtrahend, and all the signs in the subtrahend are changed.

	(14)	(15)	(16)	(17)	(18)	(19)	(20)
From	$+27a$	$-6b$	$-7c$	$+8g$	$+11h$	$-5x$	$-7y$
Take	$-13a$	$+18b$	$-c$	$-9g$	$-15h$	$+17x$	$-15y$
	$+40a$	$-24b$	$6c$	$17g$	$+26h$	$-22x$	$+8y$

In these examples we change, *mentally*, all the signs in the subtrahend, and then proceed as in addition. These questions may all be proved, as in Arithmetic, by adding the remainder to the subtrahend.

COMPOUND QUANTITIES.

69. The same rule must be observed in subtracting compound quantities, as in simple quantities; that is, all the signs in the numbers to be subtracted must be changed, the signs $+$ to $-$; and the signs $-$ to $+$; we then proceed as in addition.

(1)

$$\begin{array}{ll} \text{From} & ab+cd-ax-7 \\ \text{Take} & 4ab-3cd+4ax-15 \\ \hline & \end{array}$$

OPERATION.

$$\begin{array}{r} ab+cd-ax-7 \\ -4ab+3cd-4ax+15 \\ \hline -3ab+4cd-5ax+8 \end{array}$$

70. It is a better way for the pupil to *conceive* the signs in the subtrahend changed, but to let them remain without alteration; otherwise it might be difficult to correct errors that might arise in the operation.

(2)

$$\begin{array}{ll} \text{From} & 7x+5y-3a-6h \\ \text{Take} & x-7y+5a+11h \\ \hline & 6x+12y-8a-17h \end{array}$$

(3)

$$\begin{array}{r} 7abc-11x+5y-48 \\ 11abc+3x+7y+100 \\ \hline -4abc-14x-2y-148 \end{array}$$

(4)

$$\begin{array}{ll} \text{From} & 14h-4z+9y+x \\ \text{Take} & -3h-7z+41y-17x \\ \hline & 17h+3z-32y+18x \end{array}$$

(5)

$$\begin{array}{r} 9x-5abc-6h-51 \\ 19x-7abc-8h+1 \\ \hline 10x+2abc+2h-52 \end{array}$$

(6)

$$\begin{array}{ll} \text{From} & 3xy-a^6-3h^3-y^2 \\ \text{Take} & -xy-a^6+7h^3-10y^2 \\ \hline & 4xy-0-10h^3+9y^2 \end{array}$$

(7)

$$\begin{array}{r} 7x^2-a^3b^5+7y^5+8h^4 \\ x^2+a^3b^5-11y^5-h^4 \\ \hline 6x^2-2a^3b^5+18y^5+9h^4 \end{array}$$

(8)

$$\begin{array}{ll} \text{From} & 5\sqrt{ax^2}-3y^2+7a^{\frac{1}{2}}-1 \\ \text{Take} & 3\sqrt{ax^2}+y^2-5a^{\frac{1}{2}}+7 \\ \hline & 2\sqrt{ax^2}-4y^2+12a^{\frac{1}{2}}-8 \end{array}$$

(9)

$$\begin{array}{r} 8x^{\frac{2}{3}}+y^5+\sqrt{7h}+5 \\ 4x^{\frac{2}{3}}-3y^5-\sqrt{7h}-6 \\ \hline 4x^{\frac{2}{3}}+4y^5+2\sqrt{7h}+11 \end{array}$$

10. From $3a-5b+6h-d$ take $a+b-7d$.

Ans. $2a-6b+6h+6d$.

11. From $31x^2-3y^2+ab$ take $17x^2+5y^2-4ab+7$.

Ans. $14x^2-8y^2+5ab-7$.

12. From $5f+14b-9d$ take $-3f+7b-15d$.

Ans. $8f+7b+6d$.

13. From $11a-7b+c$ take $a+7b-3c+11$.

Ans. $10a-14b+4c-11$.

14. From m^2+3n^3 take $-4m^2-6n^3+71x$.

Ans. $5m^2+9n^3-71x$.

15. From $31a-15x-7$, take $2a-25x+y^2$.

Ans. $29a+10x-y^2-7$.

16. From abc^2-xy^3 take $-6abc^2+3xy^3-7h$.

Ans. $7abc-4xy^3+7h$.

17. From $11ch^2-5$ take $5ch^2-5+47x$.

Ans. $6ch^2-47x$.

18. From mn^2+kt take $-7mn^2+48x-y^2$.

Ans. $8mn^2+kt-48x+y^2$.

19. From $47abh-37+96y^2$ take $7abh$.

Ans. $40abh-37+96y^2$.

20. Take $7x^2y^3+hm$ from $8x^2y^3+17$.

Ans. $x^2y^3-hm+17$.

21. Take $5b^2-3c+59m$ from $11b^2$.

Ans. $6b^2+3c-59m$.

22. Take $6a-3b-5c$ from $6a+3b-5c+1$.

Ans. $6b+1$.

23. Take $41x^2+7y^3+abc$ from m^2.

Ans. $m^2-41x^2-7y^3-abc$.

24. Take x^2 from $-17x^2+14y-a+b$.

Ans. $-18x^2+14y-a+b$.

25. Take $a-b$ from $a+b$.

Ans. $+2b$.

26. From $9xz$ take $xz-7h-5m^2+7$.

Ans. $8xz+7h+5m^2-7$.

27. From $11hm+8n^2$ take x^2-y^2.

Ans. $11hm+8n^2-x^2+y^2$.

28. From $a+b$ take $a-b$, and $a-b$, and $-a+b$.

Ans. $+2b$.

29. From $a-b-c$ take $-a+b+c$ and $a-b+c$.

Ans. $a-b-3c$.

71. When similar quantities that are to be subtracted have coefficients, the operation may be performed by placing the coefficients with their proper signs within a parenthesis, and then subjoining the common quantity; thus,

$$\begin{array}{ll} \text{From} & ay-h \\ \text{Take} & dy-c \\ \hline & (a-d)y+c-h \end{array} \qquad\qquad \begin{array}{ll} \text{From} & ax^3+gy^2 \\ \text{Take} & bx^3-hy^2 \\ \hline & (a-b)x^3+(g+h)y^2 \end{array}$$

72. The subtraction of a polynomial may be indicated without performing the operation, by inclosing the quantity to be subtracted in a parenthesis, and prefixing the sign $-$.

If we wish to subtract $7a-5x+6y$, from $11a-2x+8y$, it may be indicated thus, $(11a-2x+8y)-(7a-5x+6y)$.

And $7a-3b+c+g-p$ taken from $10a$, leaves $10a-(7a-3b+c+g-p)$; being equivalent to $3a+3b-c-g+p$.

73. Algebra differs from Arithmetic in the use of negative quantities. In Algebra every quantity is either positive or negative, according as it is affected with the sign plus or minus; and, as we have observed above, whenever a quantity has not either of these signs prefixed, the sign $+$ is understood, and the quantity is said to be positive. Thus, 5 or $+5$, is positive; but -5 is negative. Positive quantities are also called affirmatives. Some Mathematicians, in treating this subject, have involved it in much perplexity, and, in our opinion, in absurdities, by considering -5, or $-a$ as quantities *less than nothing;* much to the injury, if not to the disgrace of the science. But the student is to observe that -5 denotes just the same number and quantity as $+5$, but with the additional considerations, that the former is to be *subtracted*, while the latter is to be *added*.

The simplest illustration of positive and negative quantities may be derived from a merchant's credits and debts. Five dollars are the same sum, whether it be due to him, or he owe it to another; but in one case it may be considered as *positive* $5, for it is an addition to his property; and in the other as *negative* $5, for it is subtracted from his property. And, if the sum of his debts exceed the sum of his credits by $1000, the state of his affairs may be represented by —$1000; and, undoubtedly, he is worse than if he had nothing and owed nothing. In such a case, indeed, the man is often said, in *mercantile language*, to be minus one thousand dollars. Whereas, if the sum of his credits exceed the sum of his debts by $1000, the state of his affairs may justly be represented by +$1000. These opposite signs, then, without at all affecting the absolute magnitude of the quantities to which they are prefixed, intimate the additional consideration that those quantities are in *contrary* circumstances.

SECTION IV.

MULTIPLICATION.

ART. 74. Multiplication is the repeating of a number as many times as there are units in another.

It is virtually the same in Algebra as in Arithmetic.

75. The multiplicand and multiplier may be considered as factors; and, in all operations, either may be taken for the other.

Thus, if 6 be multiplied by 7, or a by b, the result is the same, as if 7 be multiplied by 6, or b by a.

76. When several letters are written after each other, it implies that they are all multiplied by each other.

Thus, $abcd$ is the same as $a\times b\times c\times d$; and it is immaterial in what order they stand; for $abcd$, $cdab$, and $bdca$ are synonymous terms.

77. Multiplication is commonly divided into three cases.

I. When the multiplicand and multiplier are simple quantities.

II. When the multiplicand is a compound quantity, and the multiplier is a simple one.

III. When both the multiplicand and multiplier are compound quantities.

CASE I.

78. When the multiplicand and multiplier are simple quantities.

RULE. *Multiply the coefficients of both terms together, and, to the product, annex the letters in both factors, remembering that the product of like signs is plus, and, of unlike signs, is minus. That is, plus* (+) *multiplied by plus* (+), *and minus* (—) *multiplied by* (—), *give plus* (+)*; and that plus* (+) *multiplied by minus* (—), *and minus* (—) *multiplied by plus* (+) *gives minus* (—).

ILLUSTRATION.

79. 1. If a plus quantity is multiplied by a plus quantity, the result will be a plus quantity. Thus,

If $+a$ is multiplied by $+b$, it is evident that $+a$ is to be repeated as many times as there are units in $+b$; that is, b times $a=+ab$.

2. If a minus quantity is multiplied by a plus quantity, or a plus quantity by a minus quantity, the result will be a minus quantity. Thus,

If $-c$ is to be multiplied by $+d$, it is evident, that $-c$ must be repeated as many times as there are units in d; that is,

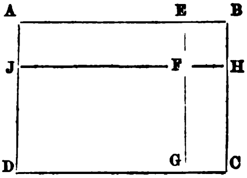
d times $-c=-cd$. The result will be the same, if $+c$ is multiplied by $-d$.

3. If a minus quantity is multiplied by a minus quantity, the result will be a plus quantity.

To illustrate this, let $a-b$ be multiplied by $c-d$.

The product of $a-b$ by c is $ac-bc$, but it is evident, that this product is as many times too large, as there are units in d. Therefore the product of $a-b$ by $d=ab-bd$, must be subtracted from $ac-bc$, thus $(ac-bc)-(ad-bd)=ac-bc-ad+bd$; but $+bd$ is the product of $-b$ and $-d$; therefore a minus quantity multiplied by a minus, is a plus quantity, *Q.E.D.*

30. That the product of two minus quantities produces a plus, may be illustrated by the following diagram.



Let $ABCD$ be a right angled parallelogram. Let JH be parallel to AB, and EG parallel to AD. Then the figure will contain four right angled parallelograms, $JFGD$, $AJFE$, $EBHF$, and $FHCG$. Let AB, which is equal to $JH=a$, and EB or its equal $FH=b$; then AE or its equal JF will be $=a-b$. Also let $AD=c$, and $AJ=d$, then JD or $FG=c-d$. Now to find the contents of $JFGD$ we must multiply the adjacent sides of the parallelogram together, which are JD and JF. But $JD=c-d$, and $JF=a-b$, therefore the contents of the parallelogram will be $(a-b)\times(c-d)=ac-ad-bc+bd$.

But ac is the contents of the figure $ABCD$, for it is the product of the adjacent sides AB and AD. And this exceeds the contents of the figure $JFGD$ by the three parallelograms $AJFE$, $EBHF$, and $FHCG$. But ad is the contents of the figure $ABHJ$, for the side $AB=a$, and $AJ=d$, and these are the adjacent sides of the parallelogram. And bc is the contents of the figure $EBCG$; for $EB=b$, and AD or $BC=c$, and therefore $bc=EBCG$, for it is the product of the adjacent sides EB and BC. But the parallelograms $ABHJ$ and $EBCG$ both include the parallelogram $EBHF$; whereas it should be included by only one of them. It must therefore be returned. The contents of this figure $EBHF=bd$; for $FH=b$, and $HB=d$, and their product is bd. And as BF has been taken twice from the figure, it is restored by considering bd a plus quantity, thus $+bd$, *Q.E.D.*

EXAMPLES.

1. Multiply $4m$ by $3n$. *Ans.* $12mn$.
2. Multiply $3ab$ by $-5cd$. *Ans.* $-15abcd$.
3. Multiply $8mn$ by $4xy$. *Ans.* $32mnxy$.
4. Multiply $7pg$ by y. *Ans.* $7pgy$.
5. Multiply $-13adef$ by $6mnp$. *Ans.* $-78adefmnp$.
6. Multiply $7hp$ by $4tuz$. *Ans.* $28hptuz$.
7. Multiply $19ab$ by $-xyz$. *Ans.* $-19abxyz$.
8. Multiply $7an$ by $-2an$. *Ans.* $-14aann$.
9. Multiply $5aaa$ by $3aaa$. *Ans.* $15aaaaaa$.
10. Multiply $-4xy$ by $-xxyy$.
11. Multiply $-11\,dc$ by $-8dee$.
12. Multiply $9mn$ by $-2mmn$.
13. Multiply $17abc$ by $-8abc$.
14. Multiply $11xyyy$ by $-yy$.
15. Multiply $-9mmm$ by $-nnn$.
16. Multiply pqt by pqt.

81. When the same letter is repeated in the product, for the sake of brevity, one letter only need be written with a figure placed before it, denoting the number of times the letter is taken.

This figure is called the *exponent* or power of the letter, and it shows how many times the letter is used as a factor. Thus, $a^3=a\times a\times a=aaa$; $4m^2=4\times m\times m=4mm$.

82. If two or more letters of the same kind, having exponents, are to be multiplied together, we write the letter, and place over it the sum of the exponents. Thus, the product of a^3 by $a^2 = aaa \times aa = aaaaa = a^5$. Hence the following

RULE. *Add the exponents of the same letter to the product of the coefficients.*

17. Multiply $4m^4$ by $3m^2$.

$$4 \times 3 \times m^4 \times m^2 = 12m^{4+2} = 12m^6.$$

18. Multiply $-5n^3$ by $-4n^5$. *Ans.* $20n^8$.
19. Multiply $-3a^m$ by $3a^m$. *Ans.* $-9a^{2m}$.
20. Multiply $2x^m$ by $4x^n$. *Ans.* $8x^{m+n}$.
21. Multiply $3a^2b^3$ by $5a^3b$. *Ans.* $15a^5b^4$.
22. Multiply ab^2 by a^3b. *Ans.* a^4b^3.
23. Multiply a^3b^2c by a^2bd. *Ans.* a^5b^3cd.
24. Multiply $7a^5c^3$ by a^4cm. *Ans.* $7a^9c^4m$.
25. Multiply $9a^5b^3x^7$ by $-a^8b^2cx^4$. *Ans.* $-9a^{13}b^5cx^{11}$.
26. Multiply $15m^5n^6$ by $3mn$. *Ans.* $45m^6n^7$.
27. Multiply $3a^mb^n$ by $2a^mb^3$. *Ans.* $6a^{2m}b^{n+3}$.
28. Multiply $4x^my^n$ by $-x^ny^nz^5$. *Ans.* $-4x^{m+n}y^{2n}z^5$.
29. Multiply $17a^4c^2$ by $4aacc$. *Ans.* $68a^6c^4$.
30. Multiply $3a^{m+n}$ by $-4a^{m-n}$.
31. Multiply $7a^m$ by $3a^{-m}$.
32. Multiply $11n^2$ by $-5n^6$.
33. Multiply $4a^6$ by $-3a^{-2}$.
34. Multiply $7m^n$ by $3m^n$.
35. Multiply $6ab^2$ by a^3b^{-4}.
36. Multiply a^{-6} by a^{-3}.
37. Multiply x^{-n} by x^n.
38. Multiply m^3 by m^{-3}.

CASE II.

83. When the multiplicand is a compound quantity, or polynomial, and the multiplier is a monomial or simple quantity.

RULE. *Multiply each term of the multiplicand, separately, by the multiplier, and prefix the proper sign to each term of the product.*

EXAMPLES.

	(1)	(2)	(3)	(4)
Multiply	$3a+5x$	$7m-4n$	$3b-4c$	$5x+7b$
By	$4m$	$3a$	$5e$	$3m$
	$12am+20mx$.	$21am-12an$.	$15be-20ce$.	$15mx+21bm$.

(5)	(6)	(7)	(8)
$4x^2-3ax^2$	$4m^3+2n$	$8a^2bc-d$	$abc+m^n$
$3x$	$3m^2$	$5ad^2$	$4am$
$12x^3-9ax^3$.	$12m^5+6m^2n$.	$40a^3bcd^2-5ad^3$.	$4a^2bcm+4am^{n+1}$.

9. Multiply $5a^2x-7y+4x^3-3b^3$ by $4ay^2$.
Ans. $20a^3xy^2-28ay^3+16ax^3y^2-12ab^3y^2$.

10. Multiply $7a^2b^3+4am^2-6y$ by $4a^5m^3$. *Ans.*

11. Multiply $4a^2b^3-6a^3c+c^2$ by $-5a^2$. *Ans.*

12. Multiply $-ab^2-3x^3-14m^{-2}$ by $-am$. *Ans.*

CASE III.

84. When both the multiplicand and multiplier are polynomials or compound quantities.

RULE. *Multiply each term of the multiplier by each term of the multiplicand, remembering, that the product of like signs is* +, *and that the product of unlike signs is* — ; *then add together all the products.*

NOTE. Terms which are alike should be placed under each other.

EXAMPLES.

(1)

$$\begin{array}{lr} \text{Multiply} & 3a+4b \\ \text{By} & 2a+b \\ \hline & 6a^2+8ab \\ & +3ab+4b^2 \\ \hline & 6a^2+11ab+4b^2. \end{array}$$

(2)

$$\begin{array}{r} x+y \\ 2x-y \\ \hline 2x^2+2xy \\ -xy-y^2 \\ \hline 2x^2+xy-y^2. \end{array}$$

(3)

$$\begin{array}{l} 7ax-4y+6m \\ 4ay+2y \\ \hline 28a^2xy-16ay^2+24amy \\ \qquad +14axy-8y^2+12my \\ \hline 28a^2xy-16ay^2+14axy-8y^2+24amy+12my. \end{array}$$

(4)

$$\begin{array}{r} 2x^2+y \\ x^2+y \\ \hline 2x^4+2x^2y \\ +2x^2y+y^2 \\ \hline 2x^4+4x^2y+y^2. \end{array}$$

(5)

$$\begin{array}{r} 3a+4m \\ 2a-2m \\ \hline 6a^2+8am \\ -6am-8m^2 \\ \hline 6a^2+2am-8m^2. \end{array}$$

(6)

$$\begin{array}{r} 3a-2b \\ 2a-5b \\ \hline 6a^2-4ab \\ -15ab+10b^2 \\ \hline 6a^2-19ab+10b^2. \end{array}$$

(7)

$$\begin{array}{l} 4a^3-5a^2b-8ab^2+2b^3 \\ 2a^2-3ab-4b^2 \\ \hline 8a^5-10a^4b-16a^3b^2+4a^2b^3 \\ \qquad -12a^4b+15a^3b^2+24a^2b^3-6ab^4 \\ \qquad\qquad -16a^3b^2+20a^2b^3+32ab^4-8b^5 \\ \hline 8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5. \end{array}$$

85. When positive and negative terms balance each other in the product, they should be cancelled.

$$\begin{array}{llll}
 & & (8) & \\
a^2 & +ax & +x^2 & \\
a & -x & & \\
\hline
a^3 & +a^2x & +ax^2 & \\
 & -a^2x & -ax^2 & -x^3 \\
\hline
a^3 & & & -x^3.
\end{array}
\qquad\qquad
\begin{array}{lllll}
 & & (9) & & \\
1 & -x & +x^2 & -x^3 & \\
1 & +x & & & \\
\hline
1 & -x & +x^2 & -x^3 & \\
 & +x & -x^2 & +x^3 & -x^4 \\
\hline
1 & & & & -x^4.
\end{array}$$

86. The continued product of factors is often expressed in one line.

10. $(1+x)\ (1+x^4)\ (1-x+x^2-x^3)=1-x^8$.

11. $(a+2x)\ (a-3x)\ (a+4x)=a^3+3a^2x-10ax^2-24x^3$.

12. Required the continued product of $3a-x$, $2a+4x$, and $4a-2x$. *Ans.* $24a^3+28a^2x-36ax^2+8x^3$.

13. Multiply $3x^2-2xy-y^2$ by $2x-4y$.
Ans. $6x^3-16x^2y+6xy^2+4y^3$.

14. Multiply x^2+2x+1 by x^2-2x+3.
Ans. x^4+4x+3.

15. Multiply $a+b-c$ by $a-b+c$.
Ans. $a^2-b^2+2bc-c^2$.

16. Multiply $3a-2b$ by $-2a+4b$.
Ans. $-6a^2+16ab-8b^2$.

17. Multiply $5a^2-3ab+4b^2$ by $6a-5b$.
Ans. $30a^3-43a^2b+39ab^2-20b^3$.

18. Multiply a^2+ab+b^2 by $a-b$. *Ans.* a^3-b^3.

19. Multiply a^4-x^4 by a^4-x^2. *Ans.* $a^8-2a^4x^4+x^8$.

20. Multiply $2x^2-3xy+6$ by $3x^2+3xy-5$.
Ans. $6x^4-3x^3y+8x^2-9x^2y^2+33xy-30$.

21. Multiply $5a^2-4ax+3x^2$ by $2a^2-3ax-4x^2$.
Ans. $10a^4-23a^3x-2a^2x^2+7ax^3-12x^4$.

22. Multiply $2a^2-3ax+4x^2$ by $5a^2-6ax-2x^2$.
Ans. $10a^4-27a^3x+34a^2x^2-18ax^3-8x^4$.

23. Multiply a^3-3a^2+3a-1 by a^2-2a+1.

Ans. $a^5-5x^4+10a^3-10a^2+5a-1$.

24. Multiply a^m-a^n by $2a-a^n$.

Ans. $2a^{m+1}-2a^{n+1}-a^{m+n}+a^{2n}$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

87. The coefficients of the polynomials should be arranged according to the successive powers of the letters increasing or decreasing by a common difference; and, when this common difference is wanting, its place should be supplied by zero.

The following examples will illustrate the above.

1. Multiply a^2+2a+1 by a^2-2a+1.

$$\begin{array}{lllll}
1 & +2 & +1 & & \\
1 & -2 & +1 & & \\ \hline
1 & +2 & +1 & & \\
 & -2 & -4 & -2 & \\
 & & +1 & +2 & +1 \\ \hline
1 & +0 & -2 & +0 & +1.
\end{array}$$

In adding the coefficients of the partial products, we perceive that the second and fourth places are a zero; but the letters must be written with their powers regularly ascending from left to right; and, where zero is the coefficient, the value of the quantity is nothing. Thus, $a^4+0a^3-2a^2+0a+1=a^4-2a^2+1$, because zero is the coefficient of the second and fourth terms.

2. Multiply x^4-x^2 by x^3+x.

$$\begin{array}{lll}
1 & -1 & \\
1 & +1 & \\ \hline
1 & -1 & \\
 & +1 & -1 \\ \hline
1 & +0 & -1.
\end{array}$$

With the letters and their powers added, it will be

$$x^7+0x^5-x^3=x^7-x^3.$$

The second term is of no value.

3. Multiply $3a^3-4ab^2+6b^3$ by $2a^2-4b^2$.

$$\begin{array}{rrrrrr}
3 & +0 & -4 & +6 & & \\
2 & +0 & -4 & & & \\ \hline
6 & +0 & -8 & +12 & & \\
 & & -12 & -0 & +16 & -24 \\ \hline
6 & +0 & -20 & +12 & +16 & -24.
\end{array}$$

We now annex the letters with their proper powers, decreasing by a constant common difference, thus:

$$6a^5+0a^4b-20a^3b^2+12a^2b^3+16ab^4-24b^5=$$
$$6a^5-20a^3b^2+12a^2b^3+16ab^4-24b^5.$$

4. Multiply $2a^3-3ab^2+5b^3$ by $2a^2-5b^2$.

$$\begin{array}{rrrrrr}
2 & +0 & -3 & +5 & & \\
2 & +0 & -5 & & & \\ \hline
4 & +0 & -6 & +10 & & \\
 & & -10 & +0 & +15 & -25 \\ \hline
4 & +0 & -16 & +10 & +15 & -25.
\end{array}$$

Affixing the letters with their powers, we have,—

$$4a^5+0a^4b-16a^3b^2+10a^2b^3+15ab^4-25b^5=$$
$$4a^5-16a^3b^2+10a^2b^3+15ab^4-25b^5.$$

5. Multiply $5a^5-3a^2+a$ by $2a^4+a^3$.

Ans. $10a^9+5a^8-6a^6-a^5+a^4$.

6. Multiply $x^3+2ax+3a^3x$ by x^2-4x+a^2. *Ans.*

7. Multiply y^2+xy-3 by y^3-y. *Ans.*

8. Multiply $7y^5+ay$ by $4y^2+y$. *Ans.*

9. Multiply $m-n+p$ by $m+n-p$.

Ans. $m^2-n^2+2np-p^2$.

10. Multiply $3a^4+3a^3b+3a^2b^2+3ab^3+3b^4$ by $7a+7b$.

Ans. $21a^5-21b^5$.

11. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Ans.* x^4-y^4.

SECTION V.

DIVISION.

ART. **88.** Division is the converse of Multiplication, and is performed, like that, of numbers. Its object is to find how many times one quantity is contained in another; or to find what quantity, multiplied by a given quantity, will produce another given quantity.

The product of like signs, as in the rule of Multiplication, produces +, and unlike signs —.

CASE I.

89. When the divisor and dividend are both simple quantities.

If abc be divided by a, the quotient will be bc, because a multiplied by bc will produce abc.

If $4abc$ be divided by $2a$, the quotient will be $2bc$, because $2a$ multiplied by $2bc$ will produce $4abc$.

If $9bx$ be divided by $3x$, the quotient is $3b$, for $3b$ multiplied by $3x$ is $9bx$.

From the above illustration we derive the following

RULE. *Write the dividend over the divisor, in the manner of a fraction, and reduce it to its simplest form by cancelling the letters and figures, that are common to each term.*

Or, divide the coefficient of the dividend by the coefficient of the divisor, and strike out from the dividend the letters common to the divisor.

EXAMPLES.

1. Divide $6ab$ by $2a$.

$$\frac{6ab}{2a}=3b;\text{ or } 6ab\div 2a=3b.$$

2. Divide $12abcxy$ by $4bx$.

$$\frac{12abcxy}{4bx}=3acy,\text{ or } 12abcxy\div 4bx=3acy.$$

3. Divide $mnop$ by op. *Ans.* mn.
4. Divide $7abm$ by am. *Ans.* $7b$.
5. Divide $14xyz$ by $7x$. *Ans.* $2yz$.
6. Divide $10abcd$ by $5\,bcd$.
7. Divide $9mnx$ by $3x$.
8. Divide $17ab$ by ab.
9. Divide $49qrst$ by $7qt$.
10. Divide $20hmno$ by $4no$.

90. Powers and roots of the same quantity are divided by subtracting the index of the divisor from that of the dividend. Thus, if we wish to divide a^5 by a^3, we subtract the index 3 from the index 5, and set the remainder 2 over the a, thus, a^2. This process is evident from the fact, that $a^5=aaaaa$, and $a^3=aaa$, and $aaaaa$ divided by aaa, gives $aa=a^2$.

11. Divide $4a^3b^4$ by $2ab^2$.

$$\frac{4a^3b^4}{2ab^2}=2a^2b^2, \text{ or } 4a^3b^4\div 2ab^2=2a^2b^2.$$

12. Divide $7a^6$ by a^2. *Ans.* $7a^4$.
13. Divide $6a^4b^2cd$ by $3ab$. *Ans.* $2a^3bcd$.
14. Divide $7r^4p^7$ by r^2p^2. *Ans.* $7r^2p^5$.
15. Divide $60p^7y^3$ by $30p^4$.
16. Divide $12ax^2y^2$ by $4ax^2$.
17. Divide $96r^4st^5u^6$ by $48st^5u^2$.
18. Divide $17a^5xy^7$ by 17.
19. Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$.

Case II.

91. When the divisor is a simple quantity, and the dividend a compound one, we adopt the following

Rule. *Divide each term of the dividend by the divisor, as in* Art. 89. *Or, we may write the divisor under the dividend, in the form of a fraction, and then cancel equal quantities, when found in each term of the divisor and dividend.*

EXAMPLES.

1. Divide $9a^3b+6a^4c-12ab$ by $3a$.

OPERATION.

$$\begin{array}{l} 3a)9a^3b+6a^4c-12ab \\ \hline \quad 3a^2b+2a^3c-4b. \ \textit{Ans.} \end{array}$$

We find, that $3a$ is a factor in each quantity, we therefore write the other factors under their respective quantities.

2. Divide $8a^3bc+16a^5bc-4a^2c^2$ by $4a^2c$. *Ans.* $2ab+4a^3b-c$.

3. Divide $9a^5bc-3a^2b+18a^3bc$ by $3ab$. *Ans.* $3a^4c-a+6a^2c$.

4. Divide $20a^4bc+15abd^3-10a^2be$ by $5ab$. *Ans.* $4a^3c+3d^3-2ae$.

5. Divide $15x^2y^3+30x^5y^7$ by x^2. *Ans.*

6. Divide $7ax^4yz^3-14xyz+21xy^2$ by $7xy$. *Ans.*

7. Divide p^2mq+p^3m-pmc by p^2. *Ans.*

8. Divide $4txz-8t^2z+z^2$ by z. *Ans.*

9. Divide $12a^{-2}-8a^2b+16a^3x-10a^{-2}y$ by $2a^2$. *Ans.* $6-4b+8ax-5y$.

CASE III.

92. When the divisor and dividend are both compound quantities.

RULE. *Write down the quantities in the same manner as in the division of numbers in Arithmetic, arranging the terms of each quantity, so that the highest powers of one of the letters may stand before the lower.*

Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient with its proper sign.

Multiply the whole divisor by the terms thus found; and having subtracted the result from the dividend, bring down as many terms to the remainder, as are requisite for the next operation, which perform as before; and so proceed, as in Arithmetic, till the work is finished.

1. Divide $a^2+2ab+b^2$ by $a+b$.

$$\begin{array}{l}
a+b)a^2+2ab+b^2(a+b.\\
\quad a^2+ab\\
\quad\quad ab+b^2\\
\quad\quad ab+b^2.
\end{array}$$

2. Divide $a^3+5a^2x+5ax^2+x^3$ by $a+x$.

$$\begin{array}{l}
a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2\\
\quad a^3+a^2x\\
\quad\quad 4a^2x+5ax^2\\
\quad\quad 4a^2x+4ax^2\\
\quad\quad\quad ax^2+x^3\\
\quad\quad\quad ax^2+x^3.
\end{array}$$

3. Divide $a^4+4a^2b^2+16b^4$ by $a^2-2ab+4b^2$.

$$\begin{array}{l}
2a^2-2ab+4b^2)a^4+4a^2b^2+16b^4(a^2+2ab+4b^2\\
\quad a^4-2a^3b+4a^2b^2\\
\quad\quad 2a^3b-16b^4\\
\quad\quad 2a^3b-4a^2b^2+8ab^3\\
\quad\quad\quad 4a^2b^2-8ab^3+16b^4\\
\quad\quad\quad 4a^2b^2+8ab^3+16b^4.
\end{array}$$

It may be verified, that $a^2+2ab+4b^2$ is the true quotient, by multiplying it by the divisor. It should also be observed, that in *every* stage of the proceeding, the terms involving the highest powers of a have been placed first on the *left*.

4. Divide $4x^4-9a^2x^2+6a^3x-a^4$ by $2x^2-3ax+a^2$.

$$\begin{array}{l}
2x^2-3ax+a^2)4x^4-9a^2x^2+6a^3x-a^4(2x^2+3ax-a^2.\\
\quad 4x^4-6ax^3+2a^2x^2\\
\quad\quad 6ax^3-11a^2x^2+6a^3x\\
\quad\quad 6ax^3-9a^2x^2+3a^3x\\
\quad\quad\quad -2a^2x^2+3a^3x-a^4\\
\quad\quad\quad -2a^2x^2+3a^3x-a^4.
\end{array}$$

93. If the divisor be not exactly contained in the dividend, the quantity that remains after the division is finished, must be placed over the divisor at the right of the quotient, in the form of a fraction.

5. Divide a^3-x^3 by $a+x$.

$$\begin{array}{l}
a+x)a^3-x^3(a^2-ax+x^2-\dfrac{2x^3}{a+x}. \\
\qquad\quad \underline{a^3-a^2x} \\
\qquad\qquad\quad a^2x-x^3 \\
\qquad\qquad\quad \underline{a^2x-ax^2} \\
\qquad\qquad\qquad\quad ax^2-x^3 \\
\qquad\qquad\qquad\quad \underline{ax^2+x^3} \\
\qquad\qquad\qquad\qquad -2x^3.
\end{array}$$

94. The operation of division may be considered as terminated, when the highest power of the letter, in the first or leading term of the remainder is less than the first term of the divisor.

The division of quantities may also be sometimes carried on *ad infinitum* like a decimal fraction; in which case, a few of the leading terms of the quotient will, generally, be sufficient to indicate the rest, without its being necessary to continue the operation.

6. Divide a by $a+x$.

$$\begin{array}{l}
a+x)a \qquad (1-\dfrac{x}{a}+\dfrac{x^2}{a^2}-\dfrac{x^3}{a^3}, \text{ \&c.} \\
\quad \underline{a+x} \\
\qquad -x \\
\quad \underline{-x-\dfrac{x^2}{a}} \\
\qquad\quad +\dfrac{x^2}{a} \\
\qquad\quad \underline{+\dfrac{x^2}{a}+\dfrac{x^3}{a^2}} \\
\qquad\qquad\quad -\dfrac{x^2}{a^2} \\
\qquad\qquad\quad \underline{-\dfrac{x^3}{a^2}-\dfrac{x^4}{a^3}.}
\end{array}$$

7. Divide a by $a-x$.

Ans. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+\frac{x^5}{a^5}$, &c.

8. Let $a^2-2ax+x^2$ be divided by $a-x$. *Ans.* $a-x$.

9. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

Ans. $a^2-2ab+b^2$.

10. Divide $8a^3-4a^2b-6ab^2+3b^3$ by $2a-b$.

Ans. $4a^2-3b^2$.

11. Divide $3b^3+3ab^2-4a^2b-4a^3$ by $a+b$.

Ans. $-4a^2+3b^2$.

12. Let $2a^2x^2-5ax+2$ be divided by $2ax-1$.

Ans. $ax-2$.

13. Divide $21a^5-21b^5$ by $7a-7b$.

Ans. $3a^4+3a^3b+3a^2b^2+3ab^3+3b^4$.

14. Divide $x^4-y^4+2y^2z^2-z^4$ by $x^2+y^2-z^2$.

Ans. $x^2-y^2+z^2$.

15. Divide $1+a$ by $1-a$.

Ans. $1+2a+2a^2+2a^3+2a^4+$, &c.

16. Divide $8x^2-15y^2+23yz-2xy-8xy-6z^2$ by $2x-3y+z$.

Ans. $4x+5y-6z$.

17. Divide $6x^4-96$ by $3x-6$.

Ans. $2x^3+4x^2+8x+16$.

DIVISION BY DETACHED COEFFICIENTS.

95. As the pupil has seen in Art. 87, that the operation of many questions in multiplication is facilitated by using detached coefficients, he will readily perceive that the same principle will apply to division.

The terms of the divisor and dividend are to be arranged according to the power of the letters, and that zero must be inserted in the terms that are wanting.

The first literal term of the quotient is obtained by dividing the first letter of the dividend by the first letter of the divisor; and the letters belonging to the other terms are written in the same order, as they are found in the divisor and dividend.

EXAMPLES.

1. Divide $a^3+3a^2b+3ab^2+b^3$ by $a+b$.

$$\begin{array}{l}
1+1)1+3+3+1(1+2+1 \text{ coefficients of the quotient.} \\
\quad\underline{1+1} \\
\qquad 2+3 \\
\qquad \underline{2+2} \\
\qquad\quad 1+1 \\
\qquad\quad \underline{1+1.}
\end{array}$$

$a^3 \div a = a^2$, first literal term of the quotient. The others will therefore be $ab+b^2$; and these terms annexed to the coefficients will be $a^2+2ab+c^2$.

2. Divide x^4-y^5 by x^2-y^2.

$$\begin{array}{l}
1+0-1)1+0+0+0-1(1+0+1 \text{ coefficients.} \\
\qquad\underline{1+0-1} \\
\qquad\qquad 1+0-1 \\
\qquad\qquad \underline{1+0-1.}
\end{array}$$

$x^4 \div x^2 = x^2$, first literal part. The other regular parts are $x^2y^2+y^4$. Having prefixed the coefficients, it will be $x^2+0xy+y^2$; but as the coefficient of the second term is zero, the term has no value. The correct answer will therefore be x^2+y^2.

3. Divide $3x^4-48$ by $3x-6$.

$$\begin{array}{l}
3-6)3+0+0+0-48(1+2+4+8 \text{ coefficients.} \\
\quad\underline{3-6} \\
\qquad 6 \\
\qquad \underline{6-12} \\
\qquad\quad 12 \\
\qquad\quad \underline{12-24} \\
\qquad\qquad 24-48 \\
\qquad\qquad \underline{24-48.}
\end{array}$$

$x^4 \div x = x^3$, first literal part. The other successive terms, will, therefore, be x^2+x+x^0. Hence the true quotient will be x^3+2x^2+4x+8.

3. Divide $1-a^8$ by $1+a$.

Ans. $1-a+a^2-a^3+a^4-a^5+a^6-a^7$.

4. Divide $3y^3+3xy^2-4x^2y-4x^3$ by $x+y$. *Ans.* $3y^2-4x^2$.

5. Divide $a^4-3a^3b-8a^2b^2+18ab^3-8b^4$ by $a^2+2ab-2b^2$.

Ans. $a^2-5ab+4b^2$.

6. Divide $m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-n^5$ by $m^2-2mn+n^2$. *Ans.* $m^3-3m^2n+3mn^2-n^3$.

7. Divide $a^{m+n}+a^{n+1}b^{m-1}-a^{m-1}b^{n+1}-b^{m+n}$ by $a^{m-1}+b^{m-1}$.

Ans. $a^{n+1}-b^{n+1}$.

SECTION VI.

FRACTIONS.

ART. **96.** Algebraic Fractions are similar to vulgar fractions in Arithmetic; they express a part or parts of a quantity or a unity.

97. They consist of two parts, the *numerator* and *denominator*, the former being written *above* the line, and the latter *below* it; and when taken together are the *terms* of the fraction.

98. The denominator shows into how many parts the quantity or unit is divided, and the numerator how many of these parts are represented by the fraction.

99. A *proper fraction* is one whose numerator is less than its denominator, as,

$$\frac{a-b}{a+d}, \text{ or } \frac{7}{8}.$$

100. An *improper fraction* is one whose numerator is equal to, or greater than its denominator, as,

$$\frac{a}{a}, \text{ or } \frac{b+c}{b-c}, \text{ or } \frac{7}{3}.$$

101. A *mixed number* is a whole number or quantity with a fraction annexed with the sign either plus or minus, as,

$$\frac{a}{b}+y,\text{ or }\frac{m}{n}-x,\text{ or }y+\frac{a}{c},\text{ or }x-\frac{m}{n},\text{ or }7\tfrac{3}{5}.$$

102. A *compound fraction* is a fraction of a fraction, connected by the word *of*, as,

$$\frac{a}{b}\text{ of }\frac{c}{d}\text{ of }\frac{m}{n},\text{ or }\tfrac{7}{8}\text{ of }\tfrac{5}{6}\text{ of }\tfrac{3}{10}.$$

103. A *complex fraction* is a fraction having a fraction or a mixed number for its numerator or denominator, or both, as,

$$\frac{\frac{3}{4}}{14\frac{1}{2}},\ \frac{\frac{4}{7}}{\frac{11}{12}},\ \frac{\frac{a}{b}}{\frac{c}{d}},\text{ or }\frac{\frac{a}{c-d}}{\frac{n}{p}}.$$

104. The value of a fraction depends on the ratio or proportion which the numerator bears to the denominator.

105. The value of a fraction is not changed by multiplying or dividing both numerator and denominator by the same number, or quantity.

106. The *greatest common measure* of two or more quantities is the largest quantity or number that will divide them without a remainder.

107. The *least common multiple* of two or more numbers or quantities is the least number or quantity that may be divided by them without a remainder.

108. A fraction is in its lowest terms, when no number or quantity, excepting a unit, will measure or divide both its terms.

109. Numbers or quantities are said to be *prime* to each other, when only a unit measures or divides them both without a remainder.

110. *Prime factors* of numbers or quantities, are those factors which can be divided by no number or quantity but by themselves or a unit; thus, the prime factors of 35 are 7 and 5.

111. A *composite quantity* is that produced by multiplying two or more quantities together.

112. A fraction is in value equal to the number of times the numerator contains the denominator.

113. A fraction is increased in value either by multiplying its numerator or dividing its denominator.

114. A fraction is said to be in its least or lowest terms, when it is expressed by the least numbers possible.

CASE I.

115. To find the greatest common measure or divisor of the terms of a fraction.

RULE. *Arrange the two quantities according to the order of their powers, and divide that which is of the highest dimensions by the other, having first cancelled any factor that may be contained in all the terms of the divisor, without being common to those of the dividend.*

Divide this divisor by the remainder, simplified, as before; and so on for each successive remainder and its preceding divisor, till nothing remains, and the last divisor will be the greatest common measure or divisor required.

If any of the divisors, in the course of the operation, become negative, they may have their signs changed, or be taken affirmatively, without altering the truth of the result; and, if the first term of a divisor should not be exactly contained in the first term of the dividend, the several terms of the latter may be multiplied by any number or quantity that will render the division complete.

EXAMPLES.

1. Find the greatest common measure or divisor of $\frac{cx+x^2}{a^2c+a^2x}$

$$cx+x^2)a^2c+a^2x$$
$$c+x)a^2c+a^2x(a^2$$
$$a^2c+a^2x.$$

As x is found in both terms of the divisor, we divide those terms by x before the operation.

The greatest common measure of both terms we perceive is $c+x$; that is, it will divide them both without a remainder. Thus, $c+x)\frac{cx+x^2}{a^2c+a^2x}=\frac{x}{a^2}$.

2. Required the greatest factor of $\frac{x^3-b^2x}{x^2+2bx+b^2}$.

$$\begin{array}{l}
x^2+2bx+b^2)x^3-b^2x(x \\
\underline{x^3+2bx^2+b^2x} \\
-2bx^2-2b^2x) \\
x+b)x^2+2bx+b^2(x+b. \\
\underline{x^2+bx} \\
bx+b^2 \\
bx+b^2
\end{array}$$

We cancel $2bx$ in both terms of the second divisor, as it is common to both.

As $x+b$ is the last divisor, it is the greatest factor or common measure of the quantities proposed.

3. Required the greatest common divisor of $3a^2-2a-1$, and $4a^3-2a^2-3a+1$.

$$\begin{array}{l}
3a^2-2a-1)4a^3-2a^2-3a+1(4a \\
\underline{3} \\
12a^3-6a^2-9a+3 \\
\underline{12a^3-8a^2-4a} \\
2a^2-5a+3)3a^2-2a-1 \\
\underline{2} \\
6a^2-4a-2(3 \\
\underline{6a^2-15a+9} \\
11a-11 \\
a-1)2a^2-5a+3(2a-3 \\
\underline{2a^2-2a} \\
-3a+3 \\
\underline{-3a+3.}
\end{array}$$

As a is common to both terms of the third divisor, it is cancelled; therefore $a-1$ is the greatest common factor of both quantities.

4. What is the greatest common divisor of x^3-a^3 and x^2-a^2? *Ans.* $x-a$.

5. What is the greatest common factor of x^2-1 and $ax+a$? *Ans.* $x-1$.

6. Required the greatest common factor of y^4-x^4 and $y^3-y^2x-yx^2+x^3$. *Ans.* y^2-x^2.

7. Required the greatest common measure of $a^3x+a^2x-ax^2-x^3$ and a^4-x^4. *Ans.* $a+x$.

8. Required the greatest common factor of a^4-x^4 and $a^5+a^3x^2$. *Ans.* a^2+x^2.

CASE II.

116. To reduce fractions to their lowest terms.

RULE. *Divide the terms of the fraction by the prime factors common to both.*

Or, divide both terms of the fraction by their greatest common divisor.

117. That fractions after reduction have the same value as before, is evident from the fact, that their numerations retain the same ratio to their denominators; for equimultiples and submultiples of any two numbers have the same ratio to each other as the numbers themselves.

Letters or numbers common to all the quantities in each term of the fraction may be cancelled.

EXAMPLES.

1. Reduce $\frac{4abc}{6a^2bd}$ to its lowest terms.

$$\frac{4abc}{6a^2bd}=\frac{2ab\times2c}{2ab\times3ad}=\frac{2c}{3ad}.\quad \textit{Ans.}$$

In this operation, we find $2ab$ to be the largest factor in both terms; it therefore may be cancelled, and the answer is $\frac{2c}{3ad}$.

2. Reduce $\frac{abxy}{admny}$ to its lowest terms.

$$\frac{abxy}{admny}=\frac{bx}{dmn}. \quad \textit{Ans.}$$

In this question, we find a and y common to both terms; and they being cancelled, the result is $\frac{bx}{dmm}$.

3. Reduce $\frac{mnopq^2}{mnop^2qx}$ to its lowest terms. *Ans.* $\frac{nq}{px}$.

4. Reduce $\frac{a^2bc}{5^2ab^2}$ to its lowest terms. *Ans.* $\frac{c}{5b}$.

5. Reduce $\frac{12am}{15bcm}$ to its lowest terms. *Ans.* $\frac{4a}{5bc}$.

6. Reduce $\frac{4ax^2}{36a^2x^3}$ to its lowest terms. *Ans.* $\frac{1}{9ax}$.

7. Reduce $\frac{6an^2}{36bn}$ to its lowest terms. *Ans.* $\frac{an}{6b}$.

8. Reduce $\frac{56bm^2x^6}{76bmx^2}$ to its lowest terms. *Ans.* $\frac{14mx^4}{19}$.

9. Reduce $\frac{19ab^2cde^3}{76a^3bcde^5}$ to its lowest terms. *Ans.* $\frac{bd^2}{4a^2e^2}$.

10. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms.

In performing this question, we first find the greatest common measure of the two terms of the fraction, which is $x+b$; we then divide both terms by it. Thus,

$$x+b)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}. \quad \textit{Ans.}$$

11. Reduce $\frac{6a^2+5ax-6x^2}{6a^2+13ax+6x^2}$ to its lowest terms.

Ans. $\frac{3a-2x}{3a+2x}$.

12. Reduce $\frac{a^2-x^2}{a^4-x^4}$ to its lowest terms. *Ans.* $\frac{1}{a^2+x^2}$.

13. It is required to reduce $\frac{x^6-y^6}{x^4-y^4}$ to its lowest terms.

Ans. $\frac{x^4+x^2y^2+y^4}{x^2+y^2}$.

CASE III.

118. To reduce a mixed quantity to the form of a fraction.

RULE. *Multiply the integral part by the denominator of the fractional part, to this product annex the numerator of the fraction, prefixing to it the sign of the fraction; under this write the denominator of the fraction.*

EXAMPLES.

1. Reduce $7\frac{3}{5}$ to a fractional form. $\frac{7\times5+3}{5}=\frac{38}{5}$. *Ans.*

2. Reduce $a+\frac{b}{e}$ to the form of a fraction.

$\frac{a\times e+b}{e}=\frac{ae+b}{e}$. *Ans.*

3. Change $a+\frac{a^2-b^2}{m}$ to a fraction.

$\frac{a\times m+\overline{a^2-b^2}}{m}=\frac{am+a^2-b^2}{m}$. *Ans.*

4. Change $a-\frac{m+n}{e}$ to the form of a fraction.

$\frac{a\times e-\overline{m+n}}{e}=\frac{ae-m-n}{e}$. *Ans.*

5. Reduce $x-\frac{a-b}{m}$ to the form of a fraction.

$\frac{x\times m-\overline{a-b}}{m}=\frac{mx-a+b}{m}$. *Ans.*

6. Change $a+\frac{b^2-cd}{n}$ to the form of a fraction.

Ans. $\frac{an+b^2-cd}{n}$.

7. Reduce $7x-\frac{4n^2+5a}{8}$ to the form of a fraction.

Ans. $\frac{56x-4n^2-5a}{8}$.

8. Reduce $15a-\frac{3m^2x-d^2}{4m}$ to the form of a fraction.

Ans. $\frac{60am-3m^2x+d^2}{4m}$.

9. Reduce $7a-b-\frac{7e-m}{4n}$ to the form of a fraction.

Ans. $\frac{28an-4bn-7e+m}{4n}$.

10. Change $11m-4n+\frac{a^2-5n^3}{3m-2n^2}$ to the form of a fraction.

Ans. $\frac{33m^2-22mn^2-12mn+3n^3}{3m-2n^2}$.

11. Reduce $8x^2+5y^2-\frac{d^3+a^2}{2x-3y^2}$ to the form of a fraction.

Ans. $\frac{16x^3-10xy^2-24x^2y^2-15y^4-d^3-a^2}{2x-3y^2}$.

Case IV.

119. To represent a fraction in the form of a whole or mixed quantity.

Rule. *Divide the numerator by the denominator for the integral part, and write the remainder, if any, over the denominator for the fractional part; annex this to the integral part, and it will represent the quantity required.*

EXAMPLES.

1. Change $\frac{27}{8}$ to a mixed quantity.

$\frac{27}{8}=27\div8=3\frac{3}{8}$. *Ans.*

2. Change $\frac{88}{11}$ to a whole number.

$\frac{88}{11}=88\div11=8$. *Ans.*

3. Change $\frac{ax+a^2}{x}$ to a mixed quantity.

$\frac{ax+a^2}{x}=\overline{ax+a^2}\div x=a+\frac{a^2}{x}$. *Ans.*

4. Change $\frac{ab-a^2}{b}$ to a mixed quantity.

$$\frac{ab-a^2}{b}=\overline{ab-a^2}\div b=a-\frac{a^2}{b}.\ \textit{Ans.}$$

5. Change $\frac{a^3-b^3+x^3}{a+x}$ to its equivalent mixed quantity.

Ans. $a^2-ax+x^2-\frac{b^3}{a+x}$.

6. Change $\frac{x^3+y^3}{x+y}$ to a whole number.

Ans. x^2-xy+y^2.

7. Change $\frac{x^3-y^3}{x-y}$ to a whole number. *Ans.* x^2+xy+y^2.

8. Find a mixed quantity equivalent to $\frac{ax^2-x}{a}$.

Ans. $x^2-\frac{x}{a}$.

CASE V.

120. To reduce a complex fraction to a simple one.

RULE. *If the numerator or denominator be whole or mixed numbers, reduce them to improper fractions. Then multiply the denominator of the lower fraction into the numerator of the upper for a new numerator, and the denominator of the upper fraction into the numerator of the lower for a new denominator; or, invert the denominator of the complex fraction when reduced, and place it in a line with the numerator; then multiply the two numerators together for a new numerator, and the two denominators together for a new denominator.*

All fractions in this proposition must be reduced to this form, $\frac{\frac{a}{c}}{\frac{d}{b}}$ or $\frac{\frac{3}{4}}{\frac{2}{5}}$, before they can be solved by the above rule. Now every fraction denotes a division of the numerator by the denominator, and its value is equal to the quotient obtained by such a division. Hence, by the nature of division, we have,

$$\frac{\frac{a}{c}}{\frac{d}{b}}=\frac{a}{c}\times\frac{b}{d}=\frac{ab}{cd}.$$

By the preceding rules we are enabled to show all the variations that can possibly happen in preparing fractions; and also the method of reducing them to their lowest terms.

EXAMPLES.

1. Reduce $\frac{\frac{7}{8}}{\frac{5}{3}}$ to a simple fraction. $\frac{\frac{7}{8}}{\frac{5}{3}}=\frac{7}{8}\times\frac{3}{5}=\frac{21}{40}$. *Ans.*

2. Reduce $\frac{7\frac{1}{4}}{8\frac{1}{2}}$ to a simple fraction.

$$\frac{7\frac{1}{4}}{8\frac{1}{2}}=\frac{\frac{29}{4}}{\frac{17}{2}}=\frac{29}{4}\times\frac{2}{17}=\frac{58}{68}=\frac{29}{34}. \quad \textit{Ans.}$$

3. Reduce $\frac{7}{\frac{1}{3}}$ to a simple fraction.

$$\frac{7}{\frac{1}{3}}=\frac{\frac{7}{1}}{\frac{1}{3}}\,\frac{7}{1}\times\frac{3}{1}=\frac{21}{1}=21. \quad \textit{Ans.}$$

4. Reduce $\frac{\frac{3}{5}}{6\frac{3}{4}}$ to a simple fraction.

$$\frac{\frac{3}{5}}{6\frac{3}{5}}=\frac{\frac{3}{5}}{\frac{27}{4}}=\frac{3}{5}\times\frac{4}{27}=\frac{12}{135}=\frac{4}{45}. \quad \textit{Ans.}$$

5. Reduce $\frac{\frac{a}{b}}{m+n}$ to a simple fraction.

$$\frac{\frac{a}{b}}{m+n}=\frac{\frac{a}{b}}{\frac{m+n}{1}}=\frac{a}{b}\times\frac{1}{m+n}=\frac{a}{bm+bn}. \quad \textit{Ans.}$$

6. Reduce $\frac{a}{x+\frac{a}{y}}=$ to a simple fraction.

$$\frac{a}{x+\frac{a}{y}}=\frac{\frac{a}{1}}{\frac{xy+a}{y}}=\frac{a}{1}\times\frac{y}{xy+a}=\frac{ay}{xy+a}. \quad \textit{Ans.}$$

7. Reduce $\dfrac{a+\dfrac{b}{c}}{x-\dfrac{m}{n}}$ to a simple fraction.

$$\frac{a+\dfrac{b}{c}}{x-\dfrac{m}{n}}=\frac{\dfrac{ac+b}{c}}{\dfrac{nx-m}{n}}=\frac{ac+b}{c}\times\frac{n}{nx-m}=\frac{acn+bn}{cnx-cm}.\ \textit{Ans.}$$

8. Reduce $\dfrac{\frac{5}{6}}{7\frac{1}{4}}$ to a simple fraction. *Ans.* $\frac{10}{87}$.

9. Reduce $\dfrac{a-\dfrac{x}{2}}{b+\dfrac{2y}{3}}$ to a simple fraction. *Ans.* $\dfrac{6a-3x}{6b+4y}$.

10. Reduce $\dfrac{m-\dfrac{n}{3}}{x}$ to a simple fraction. *Ans.* $\dfrac{3m-n}{3x}$.

11. Reduce $\dfrac{y-x+\dfrac{a}{2}}{7\frac{3}{4}}$ to a simple fraction. *Ans.* $\dfrac{8y-8x+4a}{62}$.

CASE VI.

121. To reduce fractions to a common denominator.

RULE. *Multiply each numerator into all the denominators except its own for a new numerator, and all the denominators into each other for a common denominator; or find the least common multiple of all the denominators, and it will be the denominator required.*

Divide the common multiple by each of the denominators, and multiply the quotients by the respective numerators of the fractions, and their product will be the numerators required.

1. Reduce $\frac{5}{12}$, $\frac{7}{8}$, and $\frac{3}{4}$ to a common denominator.

$$\begin{array}{l}
5\times\ 8\times4=160 \text{ numerator for } \frac{5}{12}=\frac{160}{384}.\\
7\times12\times4=336 \text{ numerator for } \frac{7}{8}=\frac{336}{384}.\\
\underline{3\times12\times8=288} \text{ numerator for } \frac{3}{4}=\frac{288}{384}.\\
12\times\ 8\times4=384 \text{ common denominator.}
\end{array}$$

Equimultiples of the terms of a fraction express the same value as the fraction itself. The terms of $\frac{5}{12}$ are each multiplied by 8 and 4. Hence $\frac{160}{384}$ has the same value as $\frac{5}{12}$. The same may be observed of $\frac{7}{8}$ and $\frac{3}{4}$.

2. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$ to a common denominator.

$$a\times d\times n=adn=\text{numerator of } \frac{a}{b}=\frac{adn}{bdn}.$$

$$c\times b\times n=bcn=\text{numerator of } \frac{c}{d}=\frac{bcn}{bdn}.$$

$$m\times b\times d=bdm=\text{numerator of } \frac{m}{n}=\frac{bdm}{bdn}.$$

$$b\times d\times n=bdn=\text{common denominator.}$$

SECOND METHOD.

3. Reduce $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{1}{4}$ to a common denominator.

$$\begin{array}{l}
4)\underline{8,\ 12,\ 4}\\
\quad 2,\ \ 3,\ 1;\ 4\times2\times3=24 \text{ common denominator.}
\end{array}$$

$$\begin{array}{r|l}
 & \underline{24}\\
8 & 3\times3=21 \text{ numerator for } \frac{7}{8}=\frac{21}{24}.\\
12 & 2\times5=10 \text{ numerator for } \frac{5}{12}=\frac{10}{24}.\\
4 & 6\times1=\ \ 6 \text{ numerator for } \frac{1}{4}=\frac{6}{24}.
\end{array}$$

4. Reduce $\frac{a}{4x}$, $\frac{b}{x^2}$, and $\frac{3a}{8x}$ to a common denominator.

$$\begin{array}{l}
x)\underline{4x,\ x^2,\ 8x}\\
4)\underline{4,\ \ x,\ \ 8}\\
\quad 1,\ \ x,\ \ 2;\ x\times4\times x\times2=8x^2 \text{ common denominator.}
\end{array}$$

$$\begin{array}{r|l} & 8x^2 \\ \cline{2-2} 4x & 2x\times a=2ax \text{ numerator to } \frac{a}{4x}=\frac{2ax}{8x^2}. \\ x^2 & 8\times b=8b \text{ numerator to } \frac{b}{x^2}=\frac{8b}{8x^2}. \\ 8x & x\times 3a=3ax \text{ numerator to } \frac{3a}{8x}=\frac{3ax}{8x^2}. \end{array}$$

5. Reduce $\frac{4}{9}$, $\frac{7}{12}$, and $\frac{1}{2}$ to a common denominator.

Ans. $\frac{16}{36}$, $\frac{21}{36}$, $\frac{18}{36}$.

6. Reduce $\frac{7}{11}$, $\frac{4}{19}$, $\frac{5}{7}$, and 7 to a common denominator.

Ans.

7. Reduce $\frac{2}{3}$ of $7\frac{1}{4}$ and $\frac{2}{11}$ of 5 to a common denominator.

Ans.

8. Reduce $\frac{3}{4}$ of $\frac{9}{11}$ of 17 and $\frac{1}{2}$ of 19 to a common denominator.

Ans.

9. Reduce $\frac{\frac{3}{8}}{\frac{5}{9}}$ and $\frac{2}{11}$ of $\frac{\frac{3}{4}}{7\frac{1}{4}}$ to a common denominator.

Ans. $\frac{8613}{12760}$, $\frac{240}{12760}$.

10. Reduce $\frac{3x}{y}$, $\frac{4m}{x}$, and $\frac{a}{b-c}$ to a common denominator.

Ans. $\frac{3bx^2-3cx^2}{bxy-cxy}$, $\frac{4bmy-4cmy}{bxy-cxy}$, $\frac{axy}{bxy-cxy}$.

11. Reduce $\frac{a}{x}$, $\frac{b}{x-2}$, and $\frac{d-3}{y}$ to a common denominator.

Ans. $\frac{axy-2ay}{x^2y-2xy}$, $\frac{bxy}{x^2y-2xy}$, $\frac{dx^2-3x^2-2dx+6x}{x^2y-2xy}$.

12. Reduce $\frac{a+b}{x}$, $\frac{3-a}{y-2}$, and $\frac{x-7}{18}$ to a common denominator.

Ans. $\frac{18ay+18by-36a-36b}{18xy-36x}$, $\frac{54x-18ax}{18xy-36x}$, $\frac{x^2y-7xy-2x^2+14x}{18xy-36x}$.

13. Reduce $\frac{4a}{b-3}$, $\frac{a}{b}$, $\frac{d}{x}$, and $\frac{a-b}{x-5}$, to a common denominator.

Ans. $\begin{cases} \dfrac{4abx^2-20abx}{b^2x^2-3bx^2-5b^2x+15bx}, & \dfrac{abx^2-3ax^2-5abx+15ax}{b^2x^2-3bx^2-5b^2x+15bx}, \\ \dfrac{b^2dx-3bdx-5b^2d+15bd}{b^2x^2-3bx^2-5b^2x+15bx}, & \dfrac{ab^2x-b^3x-3abx+3b^2x}{b^2x^2-3bx^2-5b^2x+15bx}. \end{cases}$

14. Reduce x, y, $\frac{a}{x}$, and $\frac{a}{y-3}$, to a common denominator.

Ans. $\frac{x^2y-3x^2}{xy-3x}$, $\frac{xy^2-3xy}{xy-3x}$, $\frac{ay-3a}{xy-3x}$, $\frac{ax}{xy-3x}$.

15. Reduce a, b, c, d, and $\frac{a}{b}$ to a common denominator.

Ans. $\frac{ab}{b}$, $\frac{b^2}{b}$, $\frac{bc}{b}$, $\frac{bd}{b}$, $\frac{a}{b}$.

16. Change $\dfrac{\frac{x}{y}}{\frac{m}{2}}$ and $\dfrac{\frac{x}{3y}}{\frac{1}{m-n}}$ to a common denominator.

Ans. $\frac{6x}{3my}$, and $\frac{m^2x-mnx}{3my}$.

Case VII.

ADDITION OF FRACTIONS.

122. To add fractional quantities.

Rule. *Reduce the fractions to a common denominator, and write the sum of the numerators over the common denominator.*

EXAMPLES.

1. Add $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{11}{16}$ together.

Here $$\left.\begin{array}{r} 7\times12\times16=1344 \\ 5\times8\times16=640 \\ 11\times8\times12=1056 \end{array}\right\}\text{the new numerators.}$$

$$\frac{3040}{\quad}=1\tfrac{47}{48}.\quad \textit{Ans.}$$

And $8\times12\times16=1536$ the common denominator.

2. What is the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$?

$$\text{Here } \left.\begin{aligned} a\times d\times f&=adf \\ c\times b\times f&=cbf \\ e\times b\times d&=ebd \end{aligned}\right\}\text{the new numerators.}$$

And $b\times d\times f=bdf$, the common denominator.

$$\text{Therefore, } \frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}. \quad Ans.$$

3. Add the following quantities, $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$.

$$a-\frac{3x^2}{b}=\frac{ab-3x^2}{b}\ ;\ b+\frac{2ax}{c}=\frac{bc+2ax}{c}.$$

$$\left.\begin{aligned} ab-3x^2\times c&=abc-3cx^2 \\ bc+2ax\times b&=b^2c+2abx \end{aligned}\right\}\text{numerators.}$$

$b\times c \quad =bc$, common denominator.

$$\frac{abc-3cx^2}{bc}+\frac{b^2c+2abx}{bc}=\frac{abc-3cx^2+b^2c+2abx}{bc}=a+b+\frac{2abx-3cx^2}{bc}. \quad Ans.$$

4. Add together $\frac{3a}{5d}$, $\frac{4m}{7a}$, and $\frac{3e}{4n^2}$.

$$Ans. \quad \frac{84a^2n^2+80dmn^2+105ade}{140adn^2}.$$

5. What is the sum of $\frac{7}{8}$, $\frac{5}{12}$, and $\frac{4}{9}$?

6. What is the sum of $\frac{5}{8}$, $\frac{7}{11}$, and $\frac{4}{5}$?

7. What is the sum of $\frac{8}{9}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{12}$?

8. What is the sum of $8\frac{3}{4}$, $3\frac{2}{3}$, and $7\frac{5}{6}$?

9. What is the sum of $\frac{2}{3}$ of $7\frac{1}{4}$, and $\frac{7}{11}$ of 13?

10. What is the sum of $\frac{2}{5}$ of 1, and $\frac{1}{2}$ of $\frac{2}{3}$?

11. What is the sum of $\frac{\frac{3}{4}}{7}$ and $\frac{\frac{1}{2}}{\frac{7}{11}}$?

12. What is the sum of $\frac{3}{4}$ of $\frac{4\frac{2}{3}}{11\frac{1}{2}}$ and $\frac{2}{9}$ of $\frac{\frac{3}{4}}{7\frac{2}{5}}$?

13. Find the sum of $\frac{3x}{4a}$ and $\frac{2x}{3e}$. *Ans.* $\frac{9ex+8ax}{12ae}$.

14. Find the sum of $\frac{x}{3}, \frac{x}{4}, \frac{x}{5}$. *Ans.* $\frac{47x}{60}$.

15. Find the sum of $\frac{4a}{7}$ and $\frac{a-3}{4}$. *Ans.* $\frac{23a-21}{28}$.

16. Find the sum of $4m$, $\frac{3a-1}{2}$ and $\frac{4n+2}{3}$.

Ans. $\frac{9a+24m+8n+1}{6}$.

17. What is the sum of $\frac{4a-3}{5}$, $\frac{7a+1}{3}$ and $\frac{3a}{2}$?

Ans. $\frac{139a-8}{30}$.

18. Add $\frac{3}{a+b}$ and $\frac{3}{a-b}$ together. *Ans.* $\frac{6a}{a^2-b^2}$.

19. Add $\frac{a}{a-b}$, $\frac{a-b}{c+d}$ and $\frac{a+b}{c-d}$ together.

Ans. $\frac{2a^2c+ac^2-ad^2-2abc+2abd-2a^2d}{ac^2-ad^2+bd^2-bc^2}$.

20. Add $\frac{\frac{1}{3b^2}}{\frac{a-b}{2}}$ to $\frac{\frac{b}{c}}{\frac{2a-b}{3}}$. *Ans.* $\frac{4ac-2bc+9ab^3-9b^4}{6a^2b^2c-6ab^3c-3ab^3c+3b^4c}$.

CASE VIII.

SUBTRACTION OF FRACTIONS.

ART. 123. To subtract one fractional quantity from another.

Rule. *Reduce the fractions to a common denominator, and subtract the numerator of the fraction to be subtracted from the numerator of the other fraction; under this difference write the common denominator.*

EXAMPLES.

1. From $\frac{7}{9}$ take $\frac{4}{11}$.

$$\text{Here } \left.\begin{array}{l} 7\times11=77 \\ 4\times\ 9=36 \end{array}\right\} \text{the new numerators.}$$

$$\text{And } \overline{9\times11=99}, \text{ the common denominator.}$$

Whence $\frac{77}{99}-\frac{36}{99}=\frac{41}{99}$. *Ans.*

2. From $\frac{a}{b}$ take $\frac{c}{d}$.

$$\text{Here } \left.\begin{array}{l} a\times d=ad \\ c\times b=bc \end{array}\right\} \text{the new numerators.}$$

$$\text{And } \overline{b\times d=bd}, \text{ the common denominator.}$$

Whence $\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$. *Ans.*

3. From $\frac{7}{8}$ take $\frac{4}{11}$. *Ans.* $\frac{55}{88}$.

4. From $\frac{2}{3}$ take $\frac{4}{9}$. *Ans.* $\frac{2}{9}$.

5. From $7\frac{3}{4}$ take $4\frac{1}{4}$.

6. From $6\frac{3}{11}$ take $\frac{2}{3}$ of 5.

7. From $8\frac{1}{7}$ take $\frac{2}{5}$ of $17\frac{1}{2}$.

8. From $\frac{3}{7}$ of $11\frac{2}{15}$ take $\frac{1}{12}$ of $3\frac{1}{2}$.

9. From $\frac{7}{8}$ of $13\frac{3}{7}$ take $\frac{4}{11}$ of $7\frac{1}{8}$.

10. From $\frac{3}{5}$ of 7 take $\frac{1}{9}$ of $17\frac{1}{4}$.

11. From $\dfrac{7\frac{2}{3}}{9\frac{1}{4}}$ take $\dfrac{\frac{2}{3}}{4\frac{2}{5}}$. *Ans.* $\frac{827}{1221}$.

12. Take $\dfrac{2}{a+1}$ from $\dfrac{2}{a-1}$. *Ans.* $\dfrac{4a}{a-1}$.

13. From $\dfrac{7x}{5}$ take $\dfrac{4x}{7}$. *Ans.* $\dfrac{29x}{35}$.

14. From $\frac{3a-2b}{3c}$ take $\frac{2a-4b}{5b}$.

Ans. $\frac{15ab-6ac-10b^2+12bc}{15bc}$.

15. Required the difference of $\frac{12x}{5}$ and $\frac{3x}{7}$. *Ans.* $\frac{69x}{35}$.

16. Subtract $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

17. Subtract $a-\frac{a-b}{d}$ from $3a+\frac{a+b}{d}$. *Ans.* $2a+\frac{2a}{d}$.

18. Subtract $x-\frac{4a-b}{2}$ from $7x-\frac{3a-2b}{3}$. *Ans.* $6x+a+\frac{b}{6}$.

19. From $\frac{(a+b)^2}{ab}$ take $\frac{(a-b)^2}{ab}$. *Ans.* 4.

20. From $\frac{\frac{a+b}{2}}{\frac{a-b}{3}}$ take $\frac{\frac{a-b}{3}}{\frac{a+b}{2}}$. *Ans.* $\frac{5a^2+26ab+5b^2}{6a^2-6b^2}$.

CASE IX.

MULTIPLICATION OF FRACTIONS.

124. To multiply fractional quantities together.

RULE. *Multiply the numerators together for a new numerator, and the denominators for a new denominator; and the former product being placed over the latter, will give the product of the fractions required.*

When the numerator of one of the fractions and the denominator of the other can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves.

Also, when a fraction is to be multiplied by an integer, it is *the same, whether the numerator is multiplied by it, or the denominator is divided by it.*

If an integer is to be multiplied by a fraction, or a fraction by an integer, the integer may be considered as having unity for its denominator.

A mixed quantity should be reduced to an improper fraction.

Powers or roots of the same quantity are multiplied together by adding their indices.

EXAMPLES.

1. Multiply $\frac{3}{4}$ by $\frac{7}{8}$. $\frac{3}{4}\times\frac{7}{8}=\frac{21}{32}$. *Ans.*

2. Multiply $\frac{a}{b}$ by $\frac{m}{n}$. $\frac{a}{b}\times\frac{m}{n}=\frac{am}{bn}$. *Ans.*

3. Multiply $\frac{abc}{mn}$ by $\frac{mh}{bcd}$. $\frac{abc}{mn}\times\frac{mh}{bcd}=\frac{ah}{nd}$. *Ans.*

NOTE—The b, c, and m are cancelled in both factors.

4. Multiply $\frac{3a}{7bc}$ by $2m$. $\frac{3a}{7bc}\times\frac{2m}{1}=\frac{6am}{7bc}$. *Ans.*

5. Multiply $a+\frac{hy}{a}$ by $\frac{m}{n}$.

$$a+\frac{hy}{a}=\frac{a^2+hy}{a}\,;\quad \frac{a^2+hy}{a}\times\frac{m}{n}=\frac{a^2m+hmy}{an}.$$ *Ans.*

6. Multiply $\frac{m+n}{x+y}$ by $\frac{m+n}{x+y}$.

$$\frac{m+n}{x+y}\times\frac{m+n}{x+y}=\frac{m^2+2mn+n^2}{x^2+2xy+y^2}.$$ *Ans.*

7. Multiply $\frac{a^2bc}{mn^2}$ by $\frac{a^3b^2}{m^3nd}$. *Ans.* $\frac{a^5b^3c}{m^4n^3d}$.

8. Multiply $\frac{3a^3x}{7h^4}$ by $\frac{4ab}{5hm}$. *Ans.* $\frac{12a^4bx}{35h^5m}$.

9. Multiply $\frac{5ab}{7hy}$ by $\frac{2a^3c}{3hy^3}$. *Ans.* $\frac{10a^4bc}{21h^2y^4}$.

10. Multiply $\frac{m^2n}{hy}$ by $\frac{mn^3}{ay^3}$. *Ans.* $\frac{m^3n^4}{ahy^4}$.

11. Multiply $\frac{3abx^2}{5xy^2}$ by $\frac{5xy^2}{3abx^2}$. *Ans.* 1.

When the multiplier and denominator of the fraction are the same quantity, they cancel each other.

12. Multiply $\frac{3ab}{7mm}$ by $7mn$. *Ans.* $3ab$.

13. Multiply $\frac{6mn}{11ab}$ by $11ab$. *Ans.* $6mn$.

14. Multiply $\frac{4acd}{xy}$ by xy. *Ans.* $4acd$.

15. Multiply $\frac{3hm}{17ab}$ by $17ab$. *Ans.* $3hm$.

16. Multiply $47ab$ by $\frac{x}{47ab}$. *Ans.* x.

17. Multiply $\frac{ab^{\frac{3}{4}}}{mn^{\frac{1}{2}}}$ by $\frac{b^{\frac{1}{2}}}{mn^{\frac{1}{3}}}$. *Ans.* $\frac{ab^{\frac{5}{4}}}{mn^{\frac{5}{6}}}$.

18. Multiply $\frac{mn^a}{4hy}$ by $\frac{hy}{mn^a}$. *Ans.* $\frac{1}{4}$.

19. Multiply $\frac{a^n}{b}$ by $\frac{a^m}{h}$. *Ans.* $\frac{a^{m+n}}{bh}$.

Case X.

DIVISION OF FRACTIONS.

125. To divide one fractional quantity by another.

Rule. *Multiply the denominator of the divisor by the numerator of the dividend, for the numerator; and the nu-*

merator of the divisor by the denominator of the dividend, for the denominator.

Or, invert the divisor, and proceed as in multiplication.

Or, divide the numerators by each other, and the denominators by each other, when this can be done without a remainder.

Mixed quantities should be changed to improper fractions.

EXAMPLES.

1. Divide $\frac{a}{4}$ by $\frac{3a}{8}$.

$$\frac{a}{4}\div\frac{3a}{8}=\frac{a}{4}\times\frac{8}{3a}=\frac{8a}{12a}=\frac{2}{3}.\quad \textit{Ans.}$$

2. Divide $\frac{3a}{2b}$ by $\frac{5c}{4d}$.

$$\frac{3a}{2b}\div\frac{5c}{4d}=\frac{3a}{2b}\times\frac{4d}{5c}=\frac{12ad}{10bc}=\frac{6ad}{5bc}.\quad \textit{Ans.}$$

3. Divide $\frac{2a+b}{3a-2b}$ by $\frac{3a+2b}{4a+b}$.

$$\frac{2a+b}{3a-2b}\times\frac{4a+b}{3a+2b}=\frac{8a^2+6ab+b^2}{9a^2-4b^2}.\quad \textit{Ans.}$$

4. Divide $\frac{3x}{4}$ by $\frac{11}{12}$. *Ans.* $\frac{36x}{44}=\frac{9x}{11}$.

5. Divide $\frac{6x^2}{5}$ by $3x^2$. *Ans.* $\frac{6x^2}{15x^2}=\frac{2}{5}$.

6. Divide $\frac{7m^2}{2}$ by $\frac{3n^2}{13}$. *Ans.* $\frac{91m^2}{6n^2}$.

7. Divide $\frac{11xy^2}{6}$ by 11. *Ans.* $\frac{xy^2}{6}$.

8. Divide $\frac{1}{xy}$ by xy. *Ans.* $\frac{1}{x^2y^2}$.

9. Divide $\frac{3x+1}{9}$ by $\frac{4x}{3}$. *Ans.* $\frac{3x+1}{12x}$.

10. Divide $\frac{32xy}{25}$ by $\frac{8x}{5}$. *Ans.* $\frac{4y}{5}$.

11. Divide $\frac{4x}{2x-1}$ by $\frac{2}{x+1}$. *Ans.* $\frac{2x^2+2x}{2x-1}$.

12. Divide $\frac{2a-b}{4ac}$ by $\frac{2ac}{3b-1}$. *Ans.* $\frac{-2a+b-3b^2+6ab}{8a^2c^2}$.

13. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{6a^2+5ab}{4a-4b}$.

Ans. $\frac{20a^5-20ab^4-20a^4b+20b^5}{12a^4-14a^3b-8a^2b^2+10b^4}$.

NEGATIVE EXPONENTS.

If we divide a^5 successively by a, the following will be the quotients:

$$a^4,\ a^3,\ a^2,\ a^1,\ a,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \frac{1}{a^4},\ \&c.$$

By examining the above, we perceive that the exponent of each term is one less than the preceding; therefore, the division might have been expressed thus:

$$a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4}.$$

By comparing the last quotients with the former, we find,

$$a^4=a^4;\ a^3=a^3;\ a^2=a^2;\ a=a^1;\ 1=a^0;\ \frac{1}{a}=a^{-1};\ \frac{1}{a^2}=a^{-2};$$
$$\frac{1}{a^3}=a^{-3};\ \frac{1}{a^4}=a^{-4}.$$

We also perceive that exponential quantities are divided by subtracting their indices.

Hence, if a^{-6} be divided by a^{-4}, the quotient will be $a^{-6-4}=a^{-2}$; or x^{-m} by $x^{-n}=x^{-m-n}$.

We also infer from the above illustration, that

$$\frac{a^5}{a^6},\ \frac{a^4}{a^6},\ \frac{a^3}{a^6}=a^{-1},\ a^{-2},\ a^{-3}=\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3}.$$

Again, we see from the above, that any quantity, which has zero for its exponent, is equal to 1.

We infer also, that, if similar quantities with negative exponents are divided by *subtracting* their indices, that such quantities are multiplied by *adding* their indices.

Thus, $a^{-2} \times a^{-3} = a^{-5}$, and $a^{3} \times a^{-3} = a^{3-3} = a^{0} = 1$.

EXAMPLES.

1. Divide a^{-5} by a^{-2} *Ans.* a^{-3}.
2. Divide m^{-3} by m^{-5}. *Ans.* m^{-2}.
3. Divide x^{4} by x^{-4}.
4. Divide $7x^{-2}$ by x^{-3}.
5. Divide $8y^{-2}$ by y^{2}.
6. Multiply a^{-2} by $7a^{-3}$.
7. Multiply $3m$ by m^{-4}.
8. Multiply $4x^{-1}$ by x^{0}.
9. Multiply $a^{-1}\,b^{-2}\,c^{-3}$ by $a^{4}\,b^{3}\,c^{2}$.

To free fractions from negative exponents.

RULE. *Transfer the letters which have negative exponents in the numerator to the denominator, and those which have negative exponents in the denominator to the numerator, and then change the sign of the exponent.*

EXAMPLES.

1. Free the fraction $\frac{a^{-2}\,b^{-3}}{d^{-2}\,e^{-1}}$ from negative exponents.

Ans. $\frac{d^{2}e}{a^{2}b^{3}}$.

2. Free the fraction $\frac{m\,n^{-3}\,p^{-2}}{x\,y^{-4}\,z^{-1}}$ from negative exponents.

Ans. $\frac{my^{4}z}{xn^{3}p^{2}}$.

3. Free the fraction $\frac{x^2+y^{-3}}{mn^{-4}e^{-2}}$ from negative exponents.

Ans. $\frac{e^2n^4x^2+ae^2n^4}{my^3}$.

4. Free the fraction $\frac{1-a^{-2}-y^2}{1-x^{-3}y^{-2}+x^{-2}}$ from negative exponents.

Ans. $\frac{x^3y^2+x^3y^2-x^3y^4}{a^2x^3y^2+a^2xy^2}$.

5. Free the fraction $\frac{x^{-3}y^{-2}z^{-1}}{1}$ from negative exponents.

6. Free the fraction $\frac{7a^{-3}}{a^{-1}b^{-3}c^{-4}d^{-7}e^3}$ from negative exponents.

SECTION VII.

EQUATIONS.

ART. **126.** The doctrine of equations (says Bonnycastle), is that branch of Algebra which treats of the method of determining the values of unknown quantities by means of their relations to others that are known.

This is effected by making certain algebraic expressions equal to each other, which formula, in that case, is called an *equation.*

127. The terms of an equation are the quantities of which it is composed; and the parts, that stand on each side of the sign $=$, are called the two numbers or sides of the equation.

Thus, if $x=a+b$, the terms are x, a, and b; and the meaning of the expression is, that some quantity x, standing on the left side of the equation, is equal to the sum of the quantities, a and b, on the right side.

128. A *simple equation* is that which contains only the first power of the unknown quantity; as,

$$x+a=10\,;\ ax+bx=c\,;\ \text{or } 4x+\frac{x}{4}=17\,;$$

in which equation x denotes the unknown quantity, and the other letters and numbers the known quantities.

129. A *compound equation* is that which contains two or more different powers of the unknown quantity; as $x^2+ax=d$; or $x^3-4x^2+3x=30$.

130. A *quadratic equation* is that in which the unknown quantity is of two dimensions, or which contains some letter of the second power.

131. A *cubic equation* is that in which the unknown quantity is of three dimensions, or which rises to the third power; as, $x^3=64$; or $x^3-ax^2+bx=c$.

132. The root of an equation is such a number or quantity, as being substituted for the unknown quantity, will make both sides of the equation vanish, or become equal to each other.

133. A simple equation can have but one root; but every compound one has as many roots as it has dimensions.

134. Identical equations are those which have the terms of the equation the same.

135. Numerical equations are those which contain numbers only in connection with the unknown quantities; as,

$$x^2+7x+5=100.$$

136. Literal equations are those in which numbers are represented by quantities; thus,

$$x^2+px+ap=r.$$

137. To reduce an equation is to discover the value of the unknown quantity in it, which has been represented by one of the final letters of the alphabet, or some other symbol.

138. The process of reducing equations depends upon the following simple principles or axioms:

1. If, to equal quantities, we add the same or equal quantities, the sums will be equal.

2. If, from equal quantities, we subtract the same or equal quantities, the remainders will be equal.

3. If we multiply equal quantities by the same number or quantity, the products will be equal.

4. If we divide equal quantities by the same number or quantity, the quotients will be equal.

5. If we extract the same root of equal quantities or numbers, those roots will be equal.

6. If we raise equal quantities or numbers to the same powers, those powers will be equal.

139. The known and unknown terms of an equation may be combined in various ways.

1. By addition; as, $x+7=16$, or $x+a=b$.

2. By subtraction; as $x-9=19$, or $x-a=b$.

3. By multiplication; as $7x=84$, or $ax=c$.

4. By division; as $\frac{x}{4}=12$, or $\frac{x}{a}=d$.

5. By a combination of two or more of these rules; as $\frac{3x}{4}+17=3x-5$; or, $\frac{ax}{b}+m=cx-n$.

I.

140. To find the value of the unknown quantity, when combined with a known quantity by addition or subtraction.

1. Let $x+7=16$; and it is required to find the value of x.

Now, as x plus 7 is equal to 16, it is evident from the second axiom, that, if from each of these equal quantities, we subtract the same quantity, the two remainders will be equal. We therefore subtract 7 from each member of the equation.

Thus, $x+7-7=16-7$.

As the plus 7 and minus 7, in the first member of the equation, cancel each other, the equation will be

$$x=16-7=9.$$

Therefore, the value of x is 9; but, in the operation, we have only transferred the plus 7 from the first member of the equation to the second, and changed it to a minus.

2. Again, let $x-5=12$; and it is required to find the value of x.

Now, by the first axiom, we find, if equals be added to equals, their sums will be equal; we therefore add 5 to each member of the equation, and we have

$$x-5+5=12+5.$$

In the first member of the equation, we have -5 and $+5$; and, as they will cancel each other, the equation will stand

$$x=12+5=17.$$

Therefore, the value of x is 17.

All that we virtually have done in the above operation, has been to transpose the minus 5, in the first member of the equation, to the second, and changed it to a plus.

From the foregoing examples and illustrations, we deduce the following

RULE. *When a quantity is transferred from one member of the equation to the other, the minus must be changed to a plus, and the plus to a minus.*

3. Given $x+15-5=86-8$, to find the value of x.

By transposing, $x=86-8-15+5$.
By uniting, $x=68$.

4. Given $x-29+3=100-19+3$, to find the value of x.

By transposing, $x=100-19+3+29-3$.
By uniting, $x=110$.

5. Given $x+12-3=7-4$, to find the value of x.

By transposing, $x=7-4-12+3$.
By uniting, $x=-6$.

6. Given $x-5-4=24+7$, to find the value of x.

Ans. $x=40$.

II.

141. When the known and unknown quantities are combined by multiplication.

7. What is the value of x in the equation $5x+18=58$?

By transposition, $5x=58-18$.
By reduction, $5x=40$
By division, $x=8$.

We say, that, if 5 times x is equal to 40, it is evident, that $\frac{1}{5}$ of 5 times x, that is, x, is equal to 8.

142. Hence, if the unknown quantity in any equation be multiplied by any number or quantity, in order to find its value, *we divide the sum of all the quantities, after being reduced, by the coefficient of the unknown quantity.*

8. What is the value of x in the following equation,

$$7x-28=46+10?$$

By transposition, $7x=46+10+28$.
By reduction, $7x=84$.
By division, $x=12$.

9. Given $4x-5=71+8$ to find x. *Ans.* 21.

10. Given $6x-17-7=0$ to find x. *Ans.* 4.

11. Given $5x+28+8=6$ to find x. *Ans.* -6.

12. Given $7x-17+3=100$ to find x. *Ans.* $16\frac{2}{7}$.

13. Given $23x-96+1=0$ to find x. *Ans.* $4\frac{3}{23}$.

14. Given $17x-7-5-8=4$ to find x. *Ans.* $1\frac{7}{17}$.

15. Given $9x=7+8+10$ to find x. *Ans.* $2\frac{7}{9}$.

16. Given $7x-10=5x+14$ to find x. *Ans.* 12.

III.

143. To reduce an equation, when the known and unknown terms are combined by division.

17. Given $\frac{x}{4}=8$ to find the value of x.

Multiplying both terms by 4, we have $x=32$.

If both terms of an equation be multiplied by any number, their products, by axiom third, are equal.

144. If a fraction be multiplied by its denominator, the product is the numerator, and the denominator disappears.

18. Given $\frac{3x}{5}=9$, to find the value of x.

Multiplying by 5, $3x=45$.
Dividing by 3, $x=15$.

19. Given $\frac{ax}{d}=c$, to find the value of x.

Multiplying by d, $ax=cd$.

Dividing by a, $x=\frac{cd}{a}$.

20. Given $\frac{x}{2}+\frac{2x}{3}-\frac{3x}{5}=17$, to find the value of x.

Multiplying by 2, $x+\frac{4x}{3}-\frac{6x}{5}=34$.

Multiplying by 3, $3x+4x-\frac{18x}{5}=102$.

Multiplying by 5, $15x+20x-18x=510$.

Uniting the terms, $17x=510$.

Dividing by 17, $x=30$.

145. Hence an equation may be cleared of fractions by multiplying each term of the equation by the several denominators.

21. Given $\frac{3x}{4}+\frac{5x}{6}-\frac{3x}{8}-\frac{x}{12}=9$, to find the value of x.

The least common multiple of the denominator 4, 6, 8 and 12, is 24; and multiplying each member of the equation by this number, we obtain

$$18x+20x-9x-2x=216.$$

Uniting the terms, $27x=216$.

Dividing by 27, $x=8$.

146. Hence an equation may be cleared of fractions by multiplying each term of the equation by the least common multiple of the denominators.

22. A boy being asked how many cents he had, replied, that if he had $\frac{3}{4}$ and $\frac{5}{6}$ as many in addition to what he now had, he should have 62. Required the number he had.

Let x represent the number.

Then, $\frac{3x}{4}+\frac{5x}{6}+x=62.$

By multiplying all the terms of the equation by the least common multiple of the denominators, 4 and 6, which is 12, we have

$$9x+10x+12x=744.$$

Collecting the x's, $31x=744$.

Dividing by 31, $x=24$. *Ans.*

VERIFICATION.

$$\frac{3\times24}{4}+\frac{5\times24}{6}+24=62.$$

$$18+20+24=62.$$

23. Given $\frac{15-x}{4}+3=6$ to find x.

Multiplying by 4, $15-x+12=24$.

Transposing, $15+12-24=x$.

Changing terms, $x=15+12-24$.

Reducing, $x=3$.

24. Given $\frac{5x-4}{3}-\frac{x-3}{2}=13$ to find x.

Multiplying by 3, $5x-4-\frac{3x-9}{2}=39$.

Multiplying by 2, $10x-8-3x+9=78$.

Transposing, $10x-3x=78+8-9$.

Collecting terms, $7x=77$.

Dividing by 7, $x=11$. *Ans.*

25. Given $\frac{mx-n}{a}=b$ to find x.

Multiplying by a, $mx-n=ab$.

Transposing, $mx=ab+n$.

Dividing by m, $x=\frac{ab+n}{m}$. *Ans.*

IV.

147. Combining the foregoing rules and illustrations, we deduce the following

General Rule for solving all Simple Equations, which contain only one unknown term :

1. *Clear the equation of fractions, and perform in both members all the algebraic operations required.*

2. *Transpose all the terms containing the unknown quantity to one side of the equation, and all the remaining terms to the other side, and reduce each member to its most simple form.*

3. *Divide each member of the equation by the coefficient of the unknown term.*

26. Given $2x-\frac{19}{4}=\frac{3x}{4}+4$ to find x.

Multiplying by 4, $8x-19=3x+16$.

Transposing, $8x-3x=16+19$.

Collecting, $5x=35$.

Dividing by 5, $x=7$.

27. Given $\frac{1-bx}{a}=\frac{1-ax}{b}$ to find x.

Multiplying by a, $1-bx=\frac{a-a^2x}{b}$.

Multiplying by b, $b-b^2x=a-a^2x$.

Transposing, $a^2x-b^2x=a-b$.

Dividing by a^2-b^2, $x=\frac{a-b}{a^2-b^2}=\frac{1}{a+b}$.

28. Given $\frac{a}{bx}+\frac{c}{dx}-\frac{a-c}{bdx}=h-\frac{1}{x}$ to find x.

Multiplying by bdx, $ad+bc-(a-c)=bdhx-bd$.

Omitting the parenthesis, $ad+bc-a+c=bdhx-bd$.

Changing and transposing, $bdhx=ad+bc-bd-a+c$.

Dividing by bdh, $x=\frac{ad+bc+bd-a+c}{bdh}$.

V.

148. If the terms of the equation contain both *simple* and *compound* denominators, it will, generally, be found convenient to divest it of the simple denominators at first, and, afterwards, of those which are compound.

29. Given $\frac{6x+7}{9}+\frac{7x+13}{6x+3}=\frac{2x+4}{3}$ to find x.

Multiplying by 9, $6x+7+\frac{63x+117}{6x+3}=6x+12$.

Transposing, $\frac{63x+117}{6x+3}=6x+12-6x-7=5$.

Multiplying by $6x+3$, $63x+117=30x+15$.

Transposing, $63x-30x=15-117$.

Reducing, $33x=-102$.

Dividing, $x=-3\frac{1}{11}$.

30. Given $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$ to find x.

Multiplying all the terms by 36, it being the least common multiple of 9, 3, 12, and 36, and we have

$$8x+34-\frac{468x-72}{17x-32}+12x=21x-x-16.$$

Reducing terms, $50=\frac{468x-72}{17x-32}$.

Multiplying by $17x-32$, $850x-1600=468x-72$.

Reducing terms, $382x=1528$.

Dividing by 382, $x=4$. *Ans.*

EXAMPLES.

1. Given $5x+22-2x=31$ to find x. *Ans.* $x=3$.

2. Given $4-19x=14-21x$ to find x. *Ans.* $x=5$.

3. Given $24x-12=240-12x$ to find x. *Ans.* $x=7$.

4. Given $15x+7x-10=12x+90$ to find x. *Ans.* $x=10$.

5. Given $7x+2x=12x-36$ to find x. *Ans.* $x=12$.

6. Given $12x-3x-2x=63$ to find x. *Ans.* $x=9$.

7. Given $x+\frac{x}{4}+\frac{x}{5}=87$ to find x. *Ans.* $x=60$.

8. Given $x-\frac{x}{4}+13=\frac{x}{2}+40$ to find x *Ans.* $x=108$.

9. Given $\frac{x}{5}+\frac{x}{12}=\frac{x}{10}+22$ to find x. *Ans.* $x=120$.

10. Given $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26$ to find x. *Ans.* $x=56$.

11. Given $3x+\frac{3x}{4}+15=\frac{x}{2}+41$ to find x. *Ans.* $x=8$.

12. Given $x-\frac{4x+8}{6}=8$ to find x. *Ans.* $x=28$.

13. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$ to find x. *Ans.* $x=9$.

14. Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$ to find x. *Ans.* $x=5$.

15. Given $17x-\frac{5x-4}{3}-\frac{8x+4}{5}=20x-\frac{3x+8}{2}-5$ to find x. *Ans.* $x=2$.

16. Given $9x-\frac{x-1}{2}+\frac{2x-2}{3}=12x-\frac{5x-7}{4}-13$ to find x. *Ans.* $x=7$.

17. Given $x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=2x+17$ to find x. *Ans.* $x=60$.

18. Given $\frac{a}{x}=b+c$ to find x. *Ans.* $x=\frac{a}{b+c}$.

19. Given $8x-40=0$ to find x. *Ans.* $x=5$.

20. Given $a+\frac{1}{x}=b+c+\frac{d}{x}$ to find x. *Ans.* $x=\frac{d-1}{a-b-c}$.

21. Given $5x-\frac{3x-6}{3}+4x+\frac{2x-8}{4}=8x+\frac{4x-12}{5}-\frac{3x-4}{10}+2$ to find the value of x. *Ans.* 8.

22. Given $ax^2+bx=mx^2+nx$ to find x. *Ans.* $x=\frac{n-b}{a-m}$.

23. Given $ax+m=bx+n$ to find x. *Ans.* $x=\frac{n-m}{a-b}$.

24. Given $\frac{3x}{b}-\frac{x}{c}=m-c$ to find x. *Ans.* $x=\frac{bc(m-c)}{3c-b}$.

25. Given $\frac{7x}{a}-\frac{3}{m}=15x+n$ to find x. *Ans.* $x=\frac{amn+3a}{7m-15am}$.

26. Given $\frac{a-x}{b}-\frac{4a-x}{c}=a-b$ to find x.

Ans. $x=\frac{4ab-ac+abc-b^2c}{b-c}$.

27. Given $\frac{a^2x}{b-c}+de=3x-\frac{d}{e}$ to find x.

Ans. $x=\frac{cde^2-bde^2-bd+cd}{3ce+a^2e-3be}$.

28. Given $5x-\frac{4x-a}{b}+\frac{2x+2a}{4}=m+n-\frac{2x+2a}{c}$ to find x.

Ans. $x=\frac{4bcm+4bcn-8ab-4ac-2abc}{22bc-16c+8b}$.

SECTION VIII.

PROBLEMS.

1. A GENTLEMAN stated that his age was twice that of his oldest son, and that the sum of their ages was 72 years. Required the age of each. *Ans.* 24 years, and 48 years.

2. What number is that, to which if $\frac{4}{7}$ of it be added, the sum will be 99? *Ans.* 63.

3. A and B's estate is valued at $3240; but B's is only $\frac{7}{8}$ the value of A's. What is the property of each?
Ans. A's $1728, B's $1512.

4. If $\frac{2}{3}$ of a certain number be added to $\frac{1}{2}$ of it, the sum will be 98. Required the number. *Ans.* 84.

5. I have two carriages, the value of one is five times that of the other, and the value of my horse is equal to both of my carriages. The worth of them all is $300. What is the value of each?

Ans. First carriage $25, second carriage $125; horse $150.

6. A gentleman, being asked his age, replied, that his was twice that of his wife, and that his wife was three times as old as his daughter, and that the sum of their ages was 120 years. Required the age of each.

Ans. { Gentleman's age, 72 years.
His wife's age, 36 years.
His daughter's age, 12 years.

7. A man met 4 beggars, to whom he gave 77 cents. To the first he gave twice as many as to the second; to the third, as many as he gave to the first and second; and to the fourth, as many as he gave to the first and third. What sum did he give each?

Ans. First 14 cents, second 7, third 21, fourth 35.

8. A drover has a lot of oxen and cows, for which he gave $1428. For the oxen he gave $55 each, and for the cows $32 each; and he had twice as many cows as oxen. Required the number of each. *Ans.* 12 oxen, 24 cows.

9. A gentleman at his decease left an estate of $1872 for his wife, three sons, and two daughters. His wife was to receive three times as much as either of her daughters, and his sons to receive each one half as much as one of the daughters. Required the sum each received.

Ans. Wife $864, daughters $288 each, sons $144 each.

10. A boy bought apples, oranges, and pears. He gave two cents a-piece for the apples, three cents for the oranges, and four cents for the pears; and the sum he expended was 226 cents. How many did he buy of each kind?

Ans. 21 apples, 20 oranges, and 31 pears.

11. Let 85 be divided into two such parts, that one of them shall be four times as large as the other. *Ans.* 17 and 68.

12. Divide \$100 among A, B, and C, so that A may have \$20 more than B, and B \$10 more than C.

Ans. A \$50, B \$30, and C \$20.

13. A prize of \$1000 is to be divided between A and B, so that their shares may be in the proportion of 7 to 8; required the share of each.

Ans. A's share \$466$\frac{2}{3}$, and B's \$533$\frac{1}{3}$.

14. What number is that whose 3d part exceeds its 5th part by 6$\frac{2}{5}$? *Ans.* 48.

15. A laborer agreed to serve for 36 days on these conditions, that, for every day he worked, he was to receive \$1·25, but for every day he was absent, he was to forfeit \$0·50. At the end of the time, he received \$17. It is required to find how many days he labored, and how many days he was absent.

Ans. He labored 20 days, and was absent 16 days.

16. Out of a cask of wine, which had leaked away $\frac{1}{3}$, 13 gallons were drawn, and then being gaged it was found to be half full. How many gallons did the cask contain?

Ans. 78 gallons.

17. Divide 30 into two such parts that $\frac{2}{3}$ of the one shall exceed $\frac{4}{9}$ of the other by 6$\frac{2}{3}$. *Ans.* 18 and 12.

18. What two numbers are those, whose difference is 3, and the difference of whose squares is 51? *Ans.* 10 and 7.

19. Three men, A, B, and C, trade in company; A put in a certain sum, B put in twice as much as A, and C put in three times as much as both, and they gain \$864. What is each man's share of the gain?

Ans. A's \$72, B's \$144, C's \$648.

20. James and William have between them 44 apples, and James says to William, if you will give me 12 of your apples, your number will then be only $\frac{2}{9}$ of mine. William replied, if you will give me 12 of yours, your number will then be only $\frac{3}{8}$ of mine. Required the number of each.

Ans. James had 24 apples, and William 20.

21. Let 112 be divided into two such numbers that the greater shall be the less as 9 to 7. *Ans.* 63 and 49.

22. Let 19 be divided into two such parts, that three times the greater shall be equal to four times the less. Required those numbers. *Ans.* $10\frac{6}{7}$, and $8\frac{1}{7}$.

23. There are two numbers, whose sum is 24; and, if 7 be added to the larger and 4 to the less, their ratio will be as 4 to 3. Required those numbers. *Ans.* 13 and 11.

24. The difference of two numbers is 4, and 7 times the larger number is equal to 11 times the less. Required those numbers. *Ans.* 11 and 7.

25. A merchant has two kinds of grain, one at $2·50 per bushel, and the other at $2·00 per bushel. He wishes to make a mixture of 80 bushels, that shall be worth $2·10 per bushel. How many bushels of each sort must he use?

Ans. 16 bushels at 2·50, and 64 at $2·00.

26. A man, having lost $\frac{1}{4}$ of his money, found he had $96 left. Required the sum he had at first. *Ans.* $128.

27. J. Jones found a certain sum of money, which was equal to $\frac{1}{2}$ of what he possessed, but having spent $40, the remainder was $\frac{4}{5}$ of the sum he found. Required the sum he at first possessed. *Ans.* ~~36\frac{4}{11}$~~ $57\frac{1}{7}$

28. In my school $\frac{2}{5}$ of my pupils study grammar, $\frac{2}{3}$ of the remainder read, 10 spell, and the remainder, which is $\frac{1}{7}$ of the number that read, study navigation. Required the number of pupils in the school. *Ans.* 70 pupils.

29. A gentleman lent a certain sum of money for 3 years at 5 per cent. compound interest; that is, at the end of each year he added $\frac{1}{20}$ to the sum due. At the close of the third year he lost $15·25, but then there remained due to him $2300. Required the sum lent. *Ans.* $2000.

30. A spendthrift spent $\frac{1}{5}$ of the fortune left him by his father, and he then earned $124. Soon after he lost in speculation $\frac{2}{3}$ of his property, after which he gained $274. His property was now valued at $\frac{1}{2}$, wanting $86, of his original estate. What was the sum left him by his father?

Ans. $1720.

31. A asked B how much money he had. He replied, if I had 5 times the sum I now possess, I could lend you $60, and then $\frac{1}{5}$ of the remainder would be equal to $\frac{1}{2}$ the dollars I now have. Required the sum which A had. *Ans.* $24.

32. A gentleman left an estate of $1862 for his three sons. He gave his youngest $133 less than his second son, and to his oldest son he gave as much as to the other two. How much did each receive?

Ans. Youngest son $399, second $532, oldest $931.

33. A, B, and C found a purse of money, and it was mutually agreed that A should receive $15 less than one-half, and that B should have $13 more than one-quarter, and that C should have the remainder, which was $27. How many dollars did the purse contain? *Ans.* $100.

34. Lent my good friend S. Jenkins a certain sum of money, at 6 per cent., which he kept until the interest was $\frac{3}{7}$ of the principal. The sum then due was $500. Required the sum lent. *Ans.* $350.

35. A certain man added to his estate $\frac{1}{4}$ its value, and then lost $760. But he afterwards gained $600. His property then amounted to 2000. What was the value of his estate at first? *Ans.* $1728.

36. James said to John, I have 40 shillings more than you. Yes, replied the other, and $\frac{1}{9}$ of yours is equal to $\frac{1}{4}$ of mine. Required the number of shillings that each had.

Ans. James 72 shillings, and John 32.

37. A merchant bought a number of barrels of flour, and having sold half the number and 4 barrels more to A, and $\frac{3}{4}$ of the remainder wanting 4 barrels to B, he had 20 barrels remaining. Required the number the merchant bought.

Ans. 136 barrels.

38. What number is that from which, if 7 be subtracted, $\frac{1}{6}$ of the remainder will be 5. *Ans.* 37.

39. It is required to divide 44 into two such numbers, that $\frac{3}{4}$ of one of them shall be 6 more than $\frac{3}{5}$ of the other.

Ans. 24 and 20.

40. It is required to divide the number 43 into two such parts, that one of them shall be 3 times as much above 20, as the other wants of 17. Required the numbers.

Ans. 29 and 14.

41. John Jones can reap a certain field in 10 days; but, with the help of his oldest son, he can do it in 8 days. How long would it require his son to perform the labor himself?

Ans. 40 days.

42. A engaged to reap a field for 90 shillings, and he could perform the labor in 9 days; but he took in B as a partner, and they supposed it would require 5 days for both to perform the labor, but they finished it in 4 days. How much in justice must A pay to B? *Ans.* 50 shillings.

43. I have two horses, and a saddle worth \$30. Now the saddle and first horse is worth $\frac{3}{5}$ the second horse, but the saddle and second horse is worth three times the first horse. Required the value of each.

Ans. First horse \$60, second horse \$150.

44. A gentleman let $\frac{3}{8}$ of his money at 5 per cent., and the remainder at 6 per cent.; and his interest amounted to $180. What were the sums lent?

Ans. $1200 at 5 per cent., $2000 at 6 per cent.

45. A can do a piece of work in 12 days, B can do the same work in 10 days, and C can perform it in 8 days. How long would it require A and B to do it; how long A and C; how long B and C; and how long A, B, and C to perform the labor?

Ans. A and B $5\frac{5}{11}$ days, A and C $4\frac{4}{5}$ days, B and C $4\frac{4}{9}$ days, A, B, and C $3\frac{9}{37}$ days.

46. Lent $780 at 6 per cent. for 5 years. What principal will amount to the same sum in 4 years at 10 per cent.?

Ans. $724·28$\frac{4}{7}$.

47. Lent my neighbor Jenkins $270 for 4 years at 6 per cent. Some time afterwards, I borrowed of him $500 at 8 per cent. How long shall I keep it to balance the favor?

Ans. $1\frac{31}{50}$ years.

48. A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox makes 9 leaps, while the greyhound makes but 6; but the latter, in 3 leaps, goes as far as the former in 7. How many leaps does the greyhound make before he catches the fox?

Ans. The greyhound makes 72 leaps, and the fox 108.

49. A and B engaged to reap a field for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days. They found, however, that they were obliged to call in C, an inferior workman, to assist them the last two days; in consequence of which B received 3*s*. 9*d*. less than he otherwise would have done. In what time could B and C each reap the field?

Ans. B could reap the field in 15 days, and C in 18 days.

50. A certain clock has an hour-hand, a minute-hand, and a second-hand, all turning on the same centre. At 12 o'clock, all the hands are together and point at 12. How long will it be before the second-hand will be between the other two hands, and at equal distances from each? Also, before the minute-hand will be equally distant between the other two hands? Also before the hour-hand will be equally distant between the other two hands?

Ans. $60\frac{780}{1427}$ seconds, $61\frac{683}{697}$ seconds, $59\frac{13}{73}$ seconds.

SECTION IX.

EQUATIONS OF THE FIRST DEGREE, CONTAINING TWO UNKNOWN QUANTITIES.

149. When the problem contains two unknown quantities, there must be two independent equations involving them; and from them an equation may be deduced, which shall contain only one of the unknown quantities.

The unknown quantities may be obtained by either of the following methods:

First, by Addition and Subtraction.
Second, by Comparison.
Third, by Substitution.

150. ELIMINATION BY ADDITION AND SUBTRACTION.

EXAMPLES.

1. Given $\left\{\begin{matrix} 3x-2y=11 \\ 6x+5y=67 \end{matrix}\right\}$ to find the value of x and y.

1. By first condition, $3x-2y=11.$
2. By second " $6x+5y=67.$
3. Multiplying 1st by 2, $6x-4y=22.$
4. Multiplying 2d by 1, $6x+5y=67.$
5. Subtracting 3d from 4th, $9y=45.$

6. Dividing 5th by 9, $y=5.$
7. Multiplying 1st by 5, $15x-10y=55.$
8. Multiplying 2d by 2, $12x+10y=134.$
9. Adding 7th and 8th, $27x=189.$
10. Dividing 9th by 27, $x=7.$

VERIFICATION.

$$3\times7-2\times5=21-10=11.$$
$$6\times7+5\times5=42+25=67.$$

2. Given $\left\{\begin{matrix}5x+4y=23\\6x-3y=12\end{matrix}\right\}$ to find the value of x and y.

1. By the first condition, $5x+4y=23.$
2. By the second, $6x-3y=12.$
3. Multiplying 1st by 6, $30x+24y=138.$
4. Multiplying 2d by 5, $30x-15y=60.$
5. Subtracting 4th from 3d, $39y=78.$
6. Dividing 5th by 39, $y=2.$
7. Multiplying 1st by 3, $15x+12y=69.$
8. Multiplying 2d by 4, $24x-12y=48.$
9. Adding 7th and 8th, $39x=117.$
10. Dividing 9th by 39, $x=3.$

VERIFICATION.

$$5\times3+4\times2=15+8=23.$$
$$6\times3-3\times2=18-6=12.$$

151. ELIMINATION BY ADDITION AND SUBTRACTION.

3. A says to B, if $\frac{1}{5}$ of my age be added to $\frac{2}{3}$ of yours, the sum would be $19\frac{1}{3}$ years. But, says B, if $\frac{2}{5}$ of mine were subtracted from $\frac{7}{8}$ of yours, the remainder would be $18\frac{1}{4}$ years. Required the sum of their ages.

1. By first condition, $\frac{x}{5}+\frac{2y}{3}=19\frac{1}{3}.$
2. By the second, $\frac{7x}{8}-\frac{2y}{5}=18\frac{1}{4}.$
3. Clearing the 1st of fractions, $3x+10y=290.$

4. Clearing the 2d, $35x-16y=730.$
5. Multiplying 3d by 35, $105x+350y=10150.$
6. Multiplying 4th by 3, $105x-48y=2190.$
7. Subtracting 6th from 5th, $398y=7960.$
8. Dividing 7th by 398, $y=20.$
9. Substituting 20 for y in the 3d, $3x+200=290.$
10. Transposing and uniting, $3x=90.$
11. Dividing 10th by 3, $x=30.$

VERIFICATION.

$$\frac{30}{5}+\frac{2\times20}{3}=6+13\tfrac{1}{3}=19\tfrac{1}{3}.$$

$$\frac{7\times30}{8}-\frac{2\times20}{5}=26\tfrac{1}{4}-8=18\tfrac{1}{4}.$$

From the operation of the preceding examples, we deduce the following

RULE. *Multiply or divide the given equations by such numbers or quantities as will make the term that contains one of the unknown quantities the same in each of them; then add or subtract the two equations, thus obtained, and there will arise a new equation with only one unknown quantity in it, which may be resolved by* Art. 147.

152. ELIMINATION BY COMPARISON.

4. Given $\left\{\begin{array}{l}2x+3y=17\\5x-2y=14\end{array}\right\}$ to find the values of x and y.

1. By the first condition, $2x+3y=17.$
2. By the second, $5x-2y=14.$
3. Transposition of the 1st, $2x=17-3y.$
4. Dividing the 3d by 2, $x=\dfrac{17-3y}{2}.$
5. Transposition of the 2d, $5x=14+2y.$
6. Dividing the 5th by 5, $x=\dfrac{14+2y}{5}.$

As things which are equal to the same are equal to each other, we infer, that $\frac{17-3y}{2}$, in the 4th, is equal to $\frac{14+2y}{5}$ in the 6th; because they are both equal to x.

7. Therefore, $\frac{17-3y}{2}=\frac{14+2y}{5}$.
8. Clearing of fractions, $85-15y=28+4y$.
9. Transposing 8th, $19y=57$.
10. Dividing 9th by 19, $y=3$.
11. Substituting 3 for the value of y in the first equation, we have, $2x=17-9$.
12. By reduction, $2x=8$.
13. Dividing 12th by 2, $x=4$.

VERIFICATION.

$$2\times4+3\times3=8+9=17.$$
$$5\times4-2\times3=20-6=14.$$

Hence the following

RULE. *Observe which of the unknown quantities is least involved, and find its value in each of the equations, as in* Art. 148.

Let the two values, thus found, be made equal to each other, and there will arise a new equation with only one unknown quantity in it, whose value may be found, as in Art. 147.

153. ELIMINATION BY SUBSTITUTION.

5. Two boys playing marbles, the older said to the younger, if you had three times as many marbles as you now possess, the sum of yours and mine would be 19. But the younger replied, if twice the number of mine were subtracted from four times as many as you have, the number would be 20. Required the number of marbles that each possessed.

Let x represent the marbles of the elder;

And y the number of the younger.

1. Then, by the condition of the question, $x+3y=19$,
2. And $4x-2y=20$.

3. Transposing the 1st, $x=19-3y.$
4. Putting the 3d into the 2d, $4\,(19-3y)-2y=20.$
5. Then, $76-12y-2y=20.$
6. Transposing and reducing, $y=4.$
7. Putting the value of y into the 1st, $x+12=19.$
8. Transposing and reducing, $x=19-12=7.$

Ans. The elder had 7 marbles, and the younger 4.

VERIFICATION.

$$7+3\times4=\ 7+12=19.$$
$$4\times7-2\times4=28-\ 8=20.$$

By the above method of operation, we deduce the following

Rule. *Find the value of either of the unknown quantities, in that equation in which it is least involved; then substitute this value in the place of its equal in the other equation, and there will arise a new equation with only one unknown quantity in it; the value of which may be found by* Art. 147.

EXAMPLES.

6. Given $\left\{\begin{array}{l} 3x+7y=33 \\ 2x+4y=20 \end{array}\right\}$ Required x and y.
Ans. $x=4$; $y=3$.

7. Given $\left\{\begin{array}{l} 7x+2y=39 \\ 3x-4y=\ 7 \end{array}\right\}$ Required x and y.
Ans. $x=5$; $y=2$.

8. Given $\left\{\begin{array}{l} 6x-3y=27 \\ 4x-6y=-2 \end{array}\right\}$ Required x and y.
Ans. $x=7$; $y=5$.

9. Given $\left\{\begin{array}{l} 7x+3y=62 \\ 5x-2y=36 \end{array}\right\}$ Required x and y.
Ans. $x=8$; $y=2$.

10. Given $\left\{\begin{array}{l} 12x+8y=116 \\ 2x-\ y=\ \ 3 \end{array}\right\}$ Required x and y.
Ans. $x=5$; $y=7$.

11. Given $\left\{\begin{array}{r}11x+3y=124\\ 2x-6y=-56\end{array}\right\}$ to find x and y.

Ans. $x=8$; $y=12$.

12. Given $\left\{\begin{array}{r}9x+4y=58\\ 3x+2y=26\end{array}\right\}$ to find x and y.

Ans. $x=2$; $y=10$.

13. Given $\left\{\begin{array}{r}6x+5y=112\\ 8x-2y=80\end{array}\right\}$ to find the value of x and y.

Ans. $x=12$; $y=8$.

14. Given $\left\{\begin{array}{r}7x-2y=-6\\ 2x+2y=24\end{array}\right\}$ to find the value of x and y.

Ans. $x=2$; $y=10$.

15. Given $\left\{\begin{array}{r}6x+11y=115\\ 8x-22y=-30\end{array}\right\}$ to find the value of x and y.

Ans. $x=10$; $y=5$.

16. Given $\left\{\begin{array}{r}2x+3y=47\\ 10x-12y=-62\end{array}\right\}$ to find the value of x and y.

Ans. $x=7$; $y=11$.

17. Given $\left\{\begin{array}{l}\frac{x}{2}-\frac{y}{3}=0\\ \\ \frac{x}{4}+\frac{y}{6}=6\end{array}\right\}$ to find the value of x and y.

Ans. $x=12$; $y=18$.

18. Given $\left\{\begin{array}{r}\frac{3x}{5}-y=11\\ \\ x+\frac{y}{5}=37\end{array}\right\}$ to find the value of x and y.

Ans. $x=35$; $y=10$.

19. Given $\left\{\begin{array}{r}\frac{4x}{7}-\frac{2y}{3}=2\\ x+7y=175\end{array}\right\}$ to find the value of x and y.

Ans. $x=28$; $y=21$.

20. Given $\left\{\begin{array}{l} \frac{7x}{8}-\frac{y}{9}=19 \\ 3x+3y=126 \end{array}\right\}$ to find the value of x and y.

Ans. $x=24$; $y=18$.

21. Given $\left\{\begin{array}{l} 14x+\frac{5y}{6}=38 \\ x+12y=146 \end{array}\right\}$ to find the value of x and y.

Ans. $x=2$; $y=12$.

22. Given $\left\{\begin{array}{l} \frac{x}{7}-\frac{7y}{10}=-20 \\ \frac{x}{4}+3y=134 \end{array}\right\}$ to find the value of x and y.

Ans. $x=56$; $y=40$.

23. Given $\left\{\begin{array}{l} ax+by=c \\ mx+ny=d \end{array}\right\}$ to find the value of x and y.

Ans. $x=\frac{bd-nc}{bm-an}$; $y=\frac{ad-mc}{an-bm}$.

24. Given $\left\{\begin{array}{l} \frac{x}{a}-\frac{y}{b}=m \\ \frac{x}{c}+\frac{y}{d}=n \end{array}\right\}$ to find x and y.

Ans. $x=\frac{abcm+acdn}{ad+bc}$; $y=\frac{bcdn-abdm}{ad+bc}$.

25. Given $\left\{\begin{array}{l} \frac{x}{2}-12=\frac{y}{4}+8 \\ \frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27 \end{array}\right\}$ To find the value of x and y.

Ans. $x=60$; $y=40$.

SECTION X.

ELIMINATION, WHERE THERE ARE THREE OR MORE UNKNOWN TERMS INVOLVED IN AN EQUAL NUMBER OF EQUATIONS.

Rule. *Find the values of one of the unknown quantities, in each of the three given equations, as if all the others were known; then put the first of these values equal to the second, and either of the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in* Art. 147; *and thence the value of the third.*

Or, the unknown terms may be obtained by multiplying each of the three equations by such numbers or quantities, as will make one of their terms the same in all of them; then, having subtracted any two of these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

Or, we may find the value of one of the unknown quantities in that equation in which it is least involved; and then substitute this value for that unknown quantity in all the other equations, and, proceeding in the same way with these equations, we obtain the other unknown quantities.

EXAMPLES.

1. Given $\left.\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{matrix}x+y+2z=41\\x+3y+z=47\\\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10\end{matrix}\right.\right\}$ to find the value of x, y, and z.

4. From the 1st equation, $x=41-y-2z$.

5. From the 2d, $x=47-3y-z$.

6. From the 3d, $x=20-\frac{2y}{3}-\frac{z}{2}$.

7. Equal values of x in 4th and 5th, $41-y-2z=47-3y-z$.

8. Value of y in 7th, $y=\frac{6+z}{2}$.

9. Equal values of x in 4th and 6th, $41-y-2z=20-\frac{2y}{3}-\frac{z}{2}$.

10. Value of y in 9th, $y=63-\frac{9z}{2}$.

11. Equal values of y in 8th and 10th, $\frac{6+z}{2}=63-\frac{9z}{2}$.

12. Reducing, $z=12$.

13 Substituting for z its value in 8th, $y=\frac{6+12}{2}=9$.

14. Substituting for y and z their values in 4th, $x=41-9-24=8$.

2. Given $\begin{matrix}1\\2\\3\end{matrix}\left\{\begin{array}{r}5x+4y-2z=28\\10x-6y+4z=30\\2x+\ y-\ z=\ 9\end{array}\right\}$ to find the value of x, y, and z.

Subtracting the 2d from twice the first, we have,

4. $14y-8z=26$.

Subtracting the 2d from 5 times the 3d,

5. $11y-9z=15$.

Subtracting 14 times the 5th from 11 times the 4th,

6. $38z=76$.

7. $z=2$.

Substituting for z its value in the 5th,

8. $11y-18=15$.

9. $y=3$.

Substituting for y and z their values in the 3d,

10. $2x+3-2=9$.

11. $x=4$.

3. Given $\left\{\begin{array}{r} 3x-y-2z=0 \\ 6x+2y+3z=45 \\ 4x+3y-z=31 \end{array}\right\}$ to find x, y, and z.

Ans. $x=4$; $y=6$; $z=3$.

4. Given $\left\{\begin{array}{r} 8x-9y-7z=-36 \\ 12x-y-3z=36 \\ 6x-2y-z=10 \end{array}\right\}$ to find x, y, and z.

Ans. $x=4$; $y=6$; $z=2$.

5. Given $\left\{\begin{array}{r} 7x+4y-z=78 \\ 4x-5y-3z=-21 \\ x-3y-4z=-37 \end{array}\right\}$ to find x, y, and z.

Ans. $x=8$; $y=7$; $z=6$.

6. Given $\left\{\begin{array}{l} x-y=10 \\ x+z=25 \\ y+z=15 \end{array}\right\}$ to find x, y, and z.

Ans. $x=20$; $y=10$; $z=5$.

7. Given $\left\{\begin{array}{r} 8x-4y=24-z \\ 6x+y=z+84 \\ x+80=3y+4z \end{array}\right\}$ to find x, y, and z.

Ans. $x=12$; $y=20$; $z=8$.

8. Given $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}-\frac{z}{4}=23 \\ \frac{x}{3}-\frac{y}{4}+\frac{z}{2}=12 \\ \frac{x}{4}+\frac{y}{2}-\frac{z}{3}=17 \end{array}\right\}$ to find x, y, and z.

Ans. $x=36$; $y=24$; $z=12$.

9. Given $\left\{\begin{array}{l} 3u+x+2y-z=22 \\ 4x-y+3z=35 \\ 4u+3x-2y=19 \\ 2u+4y+2z=46 \end{array}\right\}$ to find u, x, y, and z.

Ans. $u=4$; $x=5$; $y=6$; $z=7$.

EQUATIONS OF THE SECOND DEGREE CONTAINING SEVERAL UNKNOWN QUANTITIES.

EXAMPLES.

1. A says to B and C, give me half of your money and I shall have $55. B replies, if you two will give me one third of yours, I shall have $50. But C says to A and B, if I had one fifth of your money, I should have $50. Required the sum that each possessed.

Ans. A=20, B=$30, C=$40.

2. A merchant has three kinds of sugar. He can sell 3 lbs. of the first quality, 4 lbs. of the second quality, and 2 lbs. of the third quality for 60 cents; or, he can sell 4 lbs. of the first quality, 1 lb. of the second quality, and 5 lbs. of the third quality for 59 cents; or he can sell 1 lb. of the first quality, 10 lbs. of the second quality, and 3 lbs. of the third quality for 90 cents. Required the price of each quality.

Ans. First quality 8 cents per lb., second 7 cents, third 4 cents.

3. A gentleman's two horses with their harness cost him $120. The value of the worst horse with the harness was double that of the best horse; and the value of the best horse with the harness was triple that of the worst horse. What was the value of each?

Ans. Harness $50; best horse $40, worst $30.

4. Find three numbers, so that the first with half the other two, the second with one third of the other two, and the third with one fourth of the other two shall each be equal to 34.

Ans. 10, 22, and 26.

5. Find a number of three places, of which the digits have equal differences in their order; and, if the number be divided by half the sum of the digits, the quotient will be 41; and, if 396 be added to the number, the digits will be inverted.

Ans. 246.

6. A farmer has a large box filled with wheat and rye. Seven times the bushels of wheat is equal to four times the bushels of rye, wanting 3 bushels; and the quantity of wheat is to the quantity of rye as 3 to 5. Required the bushels of wheat and the bushels of rye.

Ans. Wheat 9 bushels, rye 15 bushels.

7. A says to B, if 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be $990. B replied, if 7 times my property were added to $\frac{1}{7}$ of yours, the sum would be $510. Required the property of each. *Ans.* A's $140, B's $70.

8. If $\frac{1}{7}$ of A's age were subtracted from B's age, and 5 years added to the remainder, the sum would be 6 years; and if four years were added to $\frac{1}{5}$ of B's age, it would be equal to $\frac{1}{14}$ of A's age. Required their ages.

Ans. A's 98 years, B's 15 years.

9. What fraction is that, if 1 be added to its numerator its value is $\frac{1}{3}$, or if 1 be added to its denominator its value is $\frac{1}{4}$?

Ans. $\frac{4}{15}$.

10. A says to B, if $\frac{1}{2}$ the difference of our ages were subtracted from my age, the remainder would be 25 years. B replies, if $\frac{1}{5}$ of the sum of our ages were taken from mine, the remainder would be $\frac{1}{3}$ of yours. Required their ages.

Ans. A's 30 years, B's 20 years.

11. There are two numbers, and if $\frac{1}{4}$ of their difference were taken from 4 times their sum, the remainder would be 62. But the difference of their sum and difference is equal to $\frac{2}{3}$ of the larger number. Required the numbers. *Ans.* 12 and 4.

12. Three men, reckoning their money, says the first, if $100 were added to my money, it would be as much as you both possess. Says the second, if $100 were added to my money, I should have twice as much as you two have. Says the third man, if $100 were added to mine, I should have three times as much as you both have. How much money had each man? *Ans.* First $$9\frac{1}{11}$, second $$45\frac{5}{11}$, third $63\frac{7}{11}$.

13. A, B, and C, speaking of their ages, A said that the sum of their ages was 90. B replied, that if his age were taken from the sum of the other two, the remainder would be 30. C said, if his age were taken from the other two, the remainder would be $\frac{1}{4}$ his age. Required their ages.

Ans. A's 20, B's 30, C's 40.

14. There are 4 men, A, B, C, and D, the value of whose property is $14,000. A and B's, with half of C and D's, is $9500. Half of A and B's, together with one third of C and D's, is $5500. And half of A's, with one third of B's, one fourth of C's, and one fifth of D's is $4000. Required the property of each.

Ans. A's $2000, B's $3000, C's $4000, D's $5000.

15. Find four numbers, such, that the first, together with half the second, may be 357; the second, with $\frac{1}{3}$ of the third, equal to 476; the third, with $\frac{1}{4}$ of the fourth, equal to 595; and the fourth, with $\frac{1}{5}$ of the first, equal to 714.

Ans. First number 190, second 334, third 426, fourth 676.

16. If I were to enlarge my field by making it 5 rods longer and 4 rods wider, it would contain 240 square rods more than it now does; but, if I were to make its length 4 rods less, and its breadth 5 rods less, its contents would be 210 square rods less than its present surface. What are its present length, breadth, and contents?

Ans. Length 30 rods, breadth 20 rods, contents 600 square rods.

17. A and B employ equal sums in trade. A gains $300 and B $100. Then A's stock is to B's as 4 to 3. Afterwards A gains $400 and B $200, then A's stock is to B's as 3 to 2. What sum did each employ at first in trade? *Ans.* $500.

18. A farmer having 89 oxen and cows, found, after he had sold 4 oxen and 20 cows, he had 7 more oxen than cows. What number had he of each at first?

Ans. 40 oxen, and 49 cows.

19. A and B driving their turkeys to market, A says to B, give me 5 of your turkeys, and I shall have as many as you. B replies, but give me 15 of yours, and I shall have 40 more than you. What number of turkeys had each?

Ans. A 45, and B 55 turkeys.

20. It is required to find two such numbers, that, if $\frac{1}{3}$ of the first be added to $\frac{1}{4}$ of the second, the sum shall be 25; but, if $\frac{1}{6}$ of the second be taken from $\frac{1}{4}$ of the first, the remainder will be 6. *Ans.* 48 and 36.

21. What fraction is that, if 5 be added to its numerator, its value is 2; but, if 2 be added to its denominator, its value is $\frac{1}{2}$? *Ans.* $\frac{3}{4}$.

22. B says to C, if 3 years were taken from your age and added to mine, I should be twice as old as you. C replies, if 3 years were taken from your age and added to mine, our ages would be the same. Required their ages.

Ans. B's age 21, C's age 15 years.

23. It is required to find two numbers, so that $\frac{2}{3}$ of the first added to $\frac{3}{4}$ of the second shall be $15\frac{2}{3}$, and if $\frac{1}{7}$ of the second be subtracted from $\frac{3}{4}$ of the first the remainder shall be $5\frac{1}{14}$.

Ans. 10 and 12.

24. It is required to divide 50 into two such parts, that $\frac{3}{8}$ of the larger shall be equal to $\frac{2}{3}$ of the smaller. *Ans.* 32 and 18.

25. A gentleman, at the time of his marriage, found that his wife's age was to his as 4 to 3; but, after they had been married 12 years, her age was to his as 5 to 6. Required their ages at the time of their marriage.

Ans. The man's age 24, his wife's 18 years.

26. A farmer hired a laborer for ten days, and he agreed to pay him 12 shillings for every day he labored, and he was to forfeit 8 shillings for every day he was absent; and he received at the end of his time 40 shillings. How many days did he labor, and how many days was he absent?

Ans. He labored 6 days, and was absent 4.

27. A gentleman bought a horse and chaise for $208, and $\frac{4}{7}$ of the cost of the chaise was equal to $\frac{2}{3}$ the price of the horse. What was the price of each?

Ans. Chaise $112, horse $96.

28. A can perform a piece of work in ten days; but, by the help of his son, he can perform it in 8 days. How long would it take his son to perform the work alone?

Ans. 40 days.

29. Two men, A and B, agree to dig a well in 10 days, but having labored together 4 days, B agreed to finish the job, which he did in 16 days. How long would it have required A to complete the labor? *Ans.* $9\frac{3}{5}$ days.

30. A merchant has two kinds of grain, one at 60 cents per bushel, and the other at 90 cents per bushel, of which he wishes to make a mixture of 40 bushels that may be worth 80 cents per bushel. How many bushels of each must he use?

Ans. $13\frac{1}{3}$ bushels of 60 cents, $26\frac{2}{3}$ of 90 cents.

31. A farmer has 30 bushels of oats, at 30 cents per bushel, and which he would mix with corn at 70 cents per bushel, and barley at 90 cents per bushel, so that the whole mixture may consist of 200 bushels, at 80 cents per bushel. How many bushels of corn, and how many of barley, must he mix with the oats? *Ans.* 10 bushels of corn, and 160 of barley.

32. A drover sold 6 of his oxen and 8 of his cows; and he then found he had twice as many oxen as cows. But after he had sold 10 more of his oxen, he found he had two more oxen than cows. How many had he of either at first?

Ans. 30 oxen, and 20 cows.

33. Four times the larger of two numbers is equal to six times the less, and their sum is 15. Required the numbers.

Ans. 9 and 6.

34. A and B can perform a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. In what time would each of them perform the work alone; and how long would it take them to perform the work together?

Ans. A would do the work in $9\frac{3}{5}$ days, B in 16 days, C in 48 days, A, B and C together in $5\frac{1}{3}$ days.

35. A gentleman left a sum of money to be divided among his four sons, so that the share of the oldest was $\frac{1}{2}$ of the shares of the other three; the share of the second $\frac{1}{3}$ of the sum of the other three; and the share of the third $\frac{1}{4}$ of the sum of the other three; and it was found that the share of the oldest exceeded that of the youngest by $14. What was the whole sum, and what was the share of each person?

Ans. Whole sum $120, oldest son's share $40, second son's $30, third son's $24, youngest son's $26.

SECTION XI.

NEGATIVE QUANTITIES.

Art. 154. The student will sometimes find, that, on account of his misconception of the question, that he has added a quantity which should have been subtracted, or that he has subtracted a quantity which should have been added.

This may be illustrated by the following

EXAMPLES.

1. The length of a certain field is a, and its breadth is b; how much must be added to its breadth, that its contents may be m?

Let x=the quantity to be added to its breadth.

Then $b+x=$ the breadth.

And $a(b+x)=m$ the contents.

$$ab+ax=m.$$

$$b+x=\frac{m}{a}$$

$$x=\frac{m}{a}-b.$$

2. Let the length of the field be 10 rods, and its breadth 6 rods. How many rods must be added to its breadth, that the contents of the field may be 80 square rods?

Let $x=$ the quantity to be added to the breadth. Then, by the above formula,

$$x=\frac{m}{a}-b=\frac{80}{10}-6=2 \text{ rods, the quantity to be added.}$$

VERIFICATION.

$$10\times\overline{6+2}=80 \text{ square rods.}$$

3. Let the length of the field be 10 rods, the breadth 8 rods, and it is required to find what quantity must be added to the breadth, that the contents may be 60 square rods.

By the formula,

$$x=\frac{m}{a}-b=\frac{60}{10}-8=-2 \text{ rods.}$$

We perceive by the above, that it is -2 rods, which are to be added, and not $+2$ rods; but we add quantities together in Algebra by simply writing them one after the other with their respective signs, so that -2 added to $+8$ becomes $8-2=6$, the answer, which is the same as subtracting $+2$ from $+8$. And, in general, adding a minus quantity brings the same result as subtracting a plus quantity of equal value, and *vice versa.*

VERIFICATION.

$$10\times\overline{8-2}=60 \text{ square rods. } \textit{Ans.}$$

4. Suppose the field to be 10 rods long and 8 rods wide, and it is required to ascertain how much must be subtracted from its width, that its contents may be 60 square rods.

To subtract a minus quantity is the same as adding a plus quantity. If, therefore, we change the sign of x in the formula first obtained, x will then express how much is to be subtracted.

Thus, $$-x=\frac{m}{a}-b,$$

or, $$x=b-\frac{m}{a}=8-\frac{60}{10}=2 \text{ rods.}$$

VERIFICATION.

$$10\times\overline{8-2}=60 \text{ square rods.}$$

5. If the field were 10 rods long, and 8 rods wide, how many rods must be taken from its width that its contents may be 100 square rods?

By the formula,

$$x=b-\frac{m}{a}=8-\frac{100}{10}=-2 \text{ rods.}$$

That is, minus 2 is to be subtracted from $+8$; or, as we perform subtraction in Algebra by changing the sign of the subtrahend, and thus annexing it to the minuend, we have,

$$8-(-2)=8+2=10;$$

so that, in general, subtracting a minus quantity is the same as adding a plus quantity of equal value.

6. John Smith, at the time of his marriage, was 50 years old, and his wife was 40. When will his age be twice that of his wife?

Let x=the time.

Then, $$50+x=2\times\overline{40+x},$$
$$50+x=80+2x,$$

And, $$x=50-80=-30 \text{ years.}$$

As the answer is -30 years, it is evident that he is not now twice as old as his wife, but 30 years ago his age was twice hers.

VERIFICATION.

$$50-30=\overline{40-30}\times2.$$
$$20=20.$$

7. J. Jones is 40 years old, his wife 30. When will they both be of the same age?

Let x=the time.

Then, $$40+x=30+x,$$
$$40-30=x-x,$$
And, $$10=0.$$

The answer being zero, it is certain they never will be of the same age, but that one will always be 10 years older than the other.

8. What fraction is such, that, if 2 be added to its numerator, its value is $\frac{1}{4}$; or, if 2 be added to its denominator, its value is $\frac{1}{2}$?

Ans. $\frac{-5}{-12}$.

9. What fraction is such, that, if 7 be added to the numerator, its value is nothing; but, if 2 be added to the denominator, its value is infinite?

Ans. $\frac{-7}{-2}$.

10. What fraction is that, which, if 4 be added to its numerator, its value is nothing; but, if 10 be subtracted from its denominator, its value is 1?

THE COURIERS.

1. Two couriers set out at the same time from A and C, and travel towards each other until they meet. The distance from A to C is m miles. The first courier travels a miles per hour, and the second b miles per hour. How far from A and C will they meet?

Let us suppose them to meet at B.

And let x=the distance A B,

And y=the distance B C.

Then $x+y=\text{A C}=m$.

As the first travels x miles at the rate of a miles per hour, to find the time he will travel this distance, we say, As a miles : x miles : : 1 hour : $\frac{x}{a}$=the time the first courier will travel the distance A B.

And, as b miles : x miles : : 1 hour : $\frac{x}{b}$ hours = the time the second courier will travel the distance B C.

As both couriers set out at the same time and arrive at the same time at C,

Therefore $$\frac{x}{a}=\frac{y}{b},$$

And $$x=\frac{ay}{b}.$$

If we substitute this value of x in the first equation, we have,

$$\frac{ay}{b}+y=m,$$

And $$ay+by=bm.$$

Hence $$y=\frac{bm}{a+b}.$$

Substituting this value of y in the equation $x=\frac{ay}{b}$, we have,

$$x=\frac{a}{b}\times\frac{bm}{a+b}=\frac{abm}{ab+b^2}=\frac{am}{a+b}.$$

The values of x and y in the above equation are both positive. Therefore whatever value we may assign to a, b and m, it will answer the conditions of the question.

This may be illustrated by the following question.

2. Two men, A and B, set out from two places, distant from each other 144 miles, and travel towards each other. A goes 12 miles an hour, and B four miles per hour. How far must each travel before they meet?

By the above formulæ,

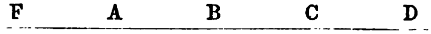
$$x=\frac{am}{a+b}=\frac{12\times144}{12+4}=108 \text{ miles, the distance A travels.}$$

And $y=\frac{bm}{a+b}=\frac{4\times 144}{12+4}=36$ miles, the distance B travels.

VERIFICATION.

$$108+36=144 \text{ miles.}$$

3. If the couriers were to set out at the same time from A and B and travel towards C, both going the same direction, the first going *a* miles per hour and the second *b* miles per hour, and the distance A B being *m*, how far would each travel before they met, suppose at a point C?



Let $x=$ the distance A C,

And $y=$ the distance B C.

Then $x-y=\text{A C}-\text{B C}=\text{A B}=m.$

By performing the same operation as in the first question, we find

$$\frac{x}{a}=\frac{y}{b},$$

$$\text{and } x=\frac{ay}{b}.$$

Therefore $\frac{ay}{b}-y=m.$

And $ay-by=bm,$

Whence $y=\frac{bm}{a-b}$

Substitute this last value of *y* in the former equation, and we have

$$x=\frac{ay}{b}=\frac{a}{b}\times\frac{bm}{a-b}=\frac{abm}{ab+b^2}=\frac{am}{a-b}.$$

Here it is evident, that the values of *x* and *y* will not be positive, unless *a* be greater than *b*. Or, in other words, unless the courier which sets out from A travels faster than the one that sets out from B, he will never overtake him.

4. Suppose the first courier to travel 9 miles per hour, and the second 6 miles per hour, and the distance AB to be 18 miles, and it was required to find how far each would travel before the one overtook the other.

Then $a=9$, $b=6$, and $m=18$.

And by the first formula,

$x=\frac{am}{a-b}=\frac{9\times18}{9-6}=54$ miles, the distance the first courier would travel.

And by the second formula,

$y=\frac{bm}{a-b}=\frac{6\times18}{9-6}=36$ miles, the distance the second courier would travel.

We perceive by the above operation, that the point C, where the couriers meet, is $54-36=18$ miles farther from A than B is, which is equal to the distance m.

5. Again, let $a=6$, $b=9$, and $m=18$; or suppose the first courier sets out from A and travels 6 miles an hour, and the second sets out at the same time from B and travels in the same direction towards C at the rate of 9 miles per hour. What distance will each travel before they meet?

By the first formula,

$$x=\frac{am}{a-b}=\frac{6\times18}{6-9}=-36 \text{ miles the first travels.}$$

By the second formula,

$$y=\frac{bm}{a-b}=\frac{9\times18}{6-9}=-54 \text{ miles the second travels.}$$

Here the value of x and y are both negative; hence there is an absolute absurdity in the question. For it is impossible that a courier, setting out from A and travelling slower than one from B, and in the same direction, should ever overtake him.

Were we to change the signs of x and y and make them negative, we should be involved in the same absurdity; for it would only show that the point where they would meet would be in an opposite direction towards F.

6. Again, let $a=6$, $b=6$, and $m=18$. Or, we will suppose the couriers both to start at the same time from A and B, and both to travel in the same direction towards C, and both travelling at the same rate of 6 miles per hour, the distance A B being 18 miles. What distance will each travel before they meet?

By the first formula,

$$x=\frac{am}{a-b}, \text{ or } \frac{am}{a-a}=\frac{am}{0},$$

or

$$x=\frac{6\times18}{6-6}=\frac{108}{0}.$$

By the second formula,

$$y=\frac{bm}{a-b}, \text{ or } \frac{bm}{a-a}=\frac{bm}{0},$$

or

$$y=\frac{6\times18}{6-6}=\frac{108}{0}.$$

As both couriers are travelling in the same direction and at the same rate, it is certain they will never meet, but the distance between them will continue the same.

155. Therefore, the expression $\frac{am}{0}$ or $\frac{108}{0}$ or any quantity with zero for a denominator, is the symbol for infinity. For it is well known that the value of a fraction depends on the number of times the numerator contains the denominator, or the number of times the denominator may be taken from the numerator until nothing shall remain.

It is certain, that, if a be greater than b, however small the difference, the couriers will eventually meet; but, if the difference between a and b be less than any assignable quantity, then x and y may be considered *infinite*.

Again, let $a=b$, and $m=0$

Then $$x=\frac{am}{a-b}=\frac{a\times 0}{0}=\frac{0}{0},$$

And $$y=\frac{bm}{a-b}=\frac{b\times 0}{0}=\frac{0}{0}.$$

From the above, we infer that x and y are equal, and that each is equal to the other.

Thus, $$x=x.$$

This is an *identical equation*, and the values of the unknown quantities cannot be known by it.

And as $m=0$, it is evident, that as both couriers start from the same point and travel at the same rate and in the same direction, they will always be together, and therefore cannot meet.

We say, therefore, that the $\frac{0}{0}$, in this case, is an expression of an Indeterminate Quantity, because that x and y may be any quantities whatever.

But it is not true, that the expression $\frac{0}{0}$ is always the sign of an indeterminate quantity.

156. In fractions, when the numerator and denominator have a common factor, and which in some cases becomes zero, and makes the fraction assume the form of $\frac{0}{0}$, but which, without that factor, has a definite value, the expression is not indeterminate.

The following fractions are examples of the kind.

$$\frac{m\,(m^2-n^2)}{n\,(m-n)}.$$

Now, if $m=n$, the value of the quantity is $\frac{0}{0}$.

But, on examination, we perceive that both the numerator and denominator have the common factor $m-n$.

Therefore, by dividing both terms of the expression by $m-n$, it becomes

$\frac{m\,(m+n)}{n}$, which, if $m=n$, is equal to $2m$.

The value of the expression

$$\frac{x-1}{x-1},$$

if we divide both terms by $x-1$, is 1; but, if $x=1$, the value is $\frac{0}{0}$.

Again, let $$x=\frac{m^3-n^3}{m-n}.$$

Then, if $m=n$, the value of $x=\frac{0}{0}$.

But, if we divide both terms by the common factor $m-n$, its value is m^2+mn+n^2, and then, on the supposition that $m=n$, its value will be $3m^2$.

INDETERMINATION.

157. In investigating the theory of indetermination, we find many curious results and apparent absurdities.

This will appear evident by investigating the following problems.

1. If it be admitted that $a=1$, and $x=1$, it may be shown, that 1 is 2, and 2 is nothing, or any assignable quantity.

Let $$a=x.$$

Multiplying both terms of the equation by x, and we have

$$ax=x^2.$$

Subtracting a^2 from both members,

$$ax-a^2=x^2-a^2.$$

Resolving both terms into factors,

$$a(x-a)=(x-a)(x+a).$$

Dividing by $x-a$,

$$a=x+a.$$

Substituting a for its value x,

$$a=a+a.$$

Dividing both terms by a,

$$1=1+1=2.$$

Again, we have found above, that

$$x^2-a^2=ax-a^2.$$

Dividing both terms by the common factor $x-a$, we have

$$x+a=\frac{ax-a^2}{x-a}.$$

Now, as x and a by the supposition are each equal to 1, we see, that

$$1+1=\frac{1\times 1-1^2}{1-1},$$

And $2=\frac{0}{0}$.

Thus it *appears*, that we have *clearly* proved, that 1 is 2, and 2 any assignable quantity. Q. E. D.

The fallacy is this, that, if nothing be divided by nothing, the quotient is any assignable quantity.

This principle may be further illustrated by considering the following identical equation.

Let $16=16$.
Resolving into terms, $12+4=12+4$.
Transposing, $4-4=12-12$.
Resolving second term into factors, $4-4=3(4-4)$.
Dividing by $4-4$, $1=3$.

Thus it appears, that 1 is 3; and in the same manner a unit may be proved to be any definite number.

From various articles in the foregoing section, we infer the following.

1. If zero be multiplied by zero, or any assignable quantity, the product is zero.

2. If zero be divided by zero, the quotient may be zero, or any assignable quantity.

3. If zero be divided by any quantity, the quotient is zero.

4. If any quantity be divided by zero, the quotient is infinity.

5. If any quantity be added to or taken from infinity, the result is infinity.

6. If zero be multiplied by infinity, the product may be any quantity.

7. If infinity be divided by infinity, the quotient may be any assignable quantity.

8. One infinity may be infinitely larger than another.

SECTION XII.

THEOREM I.

ART. **158.** If the sum of two numbers or quantities be multiplied together, their product will be equal to the squares of both numbers, plus twice their product.

NOTE.—When a number or quantity is multiplied into itself the product is a *square*.

EXAMPLES.

1. Multiply $a+b$ into itself.

$$\begin{array}{ll} a+b & \\ a+b & \\ \hline a^2+ab & \\ & ab+b^2 \\ \hline a^2+2ab+b^2 & \end{array}$$

VERIFICATION.

$$\begin{array}{lll} 8+4=12 & & 12 \\ 8+4=12 & & 12 \\ \hline 64+32 & & 144 \\ & 32+16 & \\ \hline 64+64+16=144. & & \end{array}$$

We perceive, by the above operation, that the square of any binomial may be readily obtained.

2. Multiply $3a+2b$ into itself.

$$3a\times3a+2\times3a\times2b+2b\times2b=$$
$$9a^2+12ab+4b^2.$$

3. Multiply $x+2y$ into itself. *Ans.* $x^2+4xy+4y^2$.

4. Multiply $3ab+m$ into itself. *Ans.*

5. Multiply $5y+4x$ into itself. *Ans.*

6. Multiply $2m+3n$ into itself. *Ans.*

7. Multiply $7d+2e$ into itself. *Ans.*

8. Multiply $2n+3w$ into itself. *Ans.*

9. Multiply $5a^2+2b$ into itself. *Ans.*

10. Multiply $1+\frac{1}{4}$ into itself. *Ans.*

11. Multiply $3+\frac{1}{8}$ into itself. *Ans.*

12. Multiply $2+\frac{1}{2}$ into itself. *Ans.*

THEOREM II.

159. If the sum of two numbers or quantities be multiplied by their difference, the product will be equal to the difference of their squares.

Multiply $a+b$ into $a-b$.

VERIFICATION.

$$\begin{array}{ll} a+b & \\ a-b & \\ \hline a^2+ab & \\ \quad\; -ab-b^2 & \\ \hline a^2 \qquad\; -b^2 & \end{array} \qquad \begin{array}{l} 8+4=12 \\ 8-4=\;\;4 \\ \hline 64+32 \\ \quad -32-16 \\ \hline 64 \qquad -16=48. \end{array} \qquad \begin{array}{r} 12 \\ 4 \\ \hline 48 \end{array}$$

THEOREM III.

160. If the difference of two numbers or quantities be multiplied by their difference, the product will be equal to the sum of their squares minus twice their product.

EXAMPLES.

1. Multiply $a-b$ into $a-b$.

VERIFICATION.

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2. \end{array} \qquad \begin{array}{l} 12-3=\;9 \\ 12-3=\;9 \\ \hline 144-36 \\ \qquad -36+9 \\ \hline 144-72+9=81. \end{array} \qquad \begin{array}{r} 9 \\ 9 \\ \hline 81 \end{array}$$

2. Multiply $3a-2b$ into $3a-2b$. *Ans.* $9a^2-12ab+4b^2$.

3. Multiply $5m-n$ into $5m-n$.

4. Multiply $4ab-x$ into $4ab-x$.

5. Multipiy $3a^2-b^3$ into $3a^2-b^3$.

6. Multiply x^4-y^2 into x^4-y^2.

NOTE.—If the square of the difference of two numbers be subtracted from the square of their sum, the remainder will be equal to four times their product.

$$(a+b)^2-(a-b)^2=(a^2+2ab+b^2)-(a^2-2ab+b^2)=4ab.$$

THEOREM IV.

161. If twice the product of two numbers be subtracted from the sum of their squares, the remainder will be equal to the square of their difference.

$$(a^2+b^2)-2ab=a^2-2ab+b^2.$$

But this expression, by Problem 3d, is the square of their difference.

VERIFICATION.

Let 9 and 3 be the two numbers.

Then $(9^2+3^2)-(2\times9\times3)=(9-3)^2.$

$$(81+9)-(54)=36.$$

$$90-54=36.$$

$$36=36.$$

THEOREM V.

162. If there be two numbers or quantities, one of which is divided into any number of equal parts, the product of the two quantities is equal to the product of the undivided number into the several parts of the divided number.

Let the two quantities be a and b; and let b be divided into three parts, c, d, and e.

Then $b=c+d+e,$

And $ab=ac+ad+ae.$

VERIFICATION.

Let the two numbers be 12 and 10; and let 10 be divided into the parts 5, 3, and 2.

Then $10=5+3+2,$

And $12\times10=\overline{12\times5}+\overline{12\times3}+\overline{12\times2}\,;$

$$120=60+36+24,$$

$$120=120.$$

THEOREM VI.

163. If any quantity or number be divided into two parts, the square of this number or quantity will be equal to the sum of the products of this number into its two parts.

Let a represent the quantity, and b and c the parts into which it is divided.

Then $a=b+c$,

And $a\times a=a(b+c)$,

$$a^2=ab+ac.$$

VERIFICATION.

Let 12 be divided into two parts, 9 and 3.

Then $12=9+3$,

$$12\times 12=12(9+3),$$

$$144=108+36=144.$$

THEOREM VII.

164. If any quantity or number be divided into two parts, the product of the whole and one of the parts will be equal to the product of the two parts plus the square of the aforesaid part.

Let a represent the whole quantity, and b and c the parts.

Then $a=b+c$.

Multiplying both sides of the equation by b, we have,

$$ab=b^2+bc.$$

VERIFICATION.

Let 12 represent the number, and 9 and 3 the parts into which it is divided.

Then $12=9+3$.

Multiplying both parts of the equation by 9, we have,

$$9\times 12=9(9+3),$$

$$108=81+27=108.$$

THEOREM VIII.

165. If any quantity or number be divided into two parts, the square of the whole quantity or number will be equal to the squares of the two parts plus twice their product.

Let a represent any quantity, and b and c the parts into which it is divided.

Then $$a=b+c.$$

By squaring both sides of the equation we have,

$$a^2=b^2+2bc+c^2.$$

VERIFICATION.

Let 9 be divided into two parts, 6 and 3.

Then $$9=6+3.$$

By squaring both parts of the equation we have,

$$9^2=(6+3)^2,$$
$$81=36+36+9=81.$$

THEOREM IX.

166. If any number or quantity be divided into two equal parts, and into two unequal parts, the square of one of the equal parts will be equal to the product of the two unequal parts, plus the square of half the difference of the two unequal parts.

Let a represent one of the equal parts, and b and c the two unequal parts.

Then $$a=\frac{b+c}{2};$$

And $$2a=b+c;$$
$$4a^2=b^2+2bc+c^2.$$

We now add $-4bc$ to both sides of the equation.

And $$4a^2-4bc=-4bc+b^2+2bc+c^2;$$
$$4a^2-4bc=b^2-2bc+c^2;$$
$$a^2-bc=\frac{b^2-2bc+c^2}{4};$$
$$a^2=bc+\frac{b^2-2bc+c^2}{4}.$$

VERIFICATION.

Let 12 be divided into two equal parts, 6 and 6; and into two unequal parts, 9 and 3.

Then $6^2=9\times3+\frac{9^2-2\times9\times3+3^2}{4}$;

And $36=27+\frac{81-54+9}{4}$;

$$36=27+9=36.$$

THEOREM X.

167. If any quantity $2a$ be divided into two equal parts, and if any quantity b be added to $2a$, the product of $2a+b$ into b, plus the square of a, is equal to the square of $2a+b$. Then by the proposition, $2a+b$ will be the whole quantity.

Multiplying by b, we have

$$b(2a+b)=2ab+b^2.$$

By adding a^2 to each member of the equation, we have

$$b(2a+b)+a^2=a^2+2ab+b^2;$$

Therefore $b(2a+b)+a^2=(a+b)^2$.

VERIFICATION.

Let $a=10$, and $b=2$.

Then $2(2\times10+2)+10^2=(10+2)^2$;

$$144 \qquad 144$$

THEOREM XI.

168. If any quantity or number be divided into two parts, the square of this quantity or number, and one of the parts, will be equal to twice the product of the whole quantity and that part, plus the square of the other part.

Let the whole quantity be denoted by a, and the parts by b and c.

Then $a=b+c$;

And $a-c=b$;

$$a^2-2ac+c^2=b^2;$$

$$a^2+c^2=2ac+b^2.$$

VERIFICATION.

Let $12=3+9$.

Then $12^2+9^2=2\times12\times9+3^2$;

And $144+81=216+9$;

$$225=225.$$

THEOREM XII.

169. If any quantity or number be divided into any two parts, four times the product of the whole quantity into one of the parts, plus the square of the other part, will be equal to the square of the quantity, which consists of the whole and the first mentioned part.

Let a represent the quantity, and b and c the two parts into which it is divided.

Then $$a=b+c.$$

Multiplying both members of the equation by $4b$, we shall have
$$4b\times a=4b\times(b+c);$$
$$4ab=4b^2+4bc.$$

We now add c^2 to both members.
$$4ab+c^2=c^2+4bc+4b^2;$$
or $$4ab+c^2=(c+2b)^2.$$

VERIFICATION.

Let $a=12$, and $b=9$, and $c=3$.

Then $$12=9+3;$$

And $$4\times12\times9+3^2=(3\times\overline{2\times9})^2;$$
$$441 = 441.$$

THEOREM XIII.

170. If any quantity or number be divided into two equal parts, and also into two unequal parts, the sum of the squares of the two unequal parts will be double the squares of half the quantity, plus twice the square of the quantity, which consists of the difference between half the quantity and the larger of the unequal parts of the quantity.

Let $2a$ represent the quantity, and $a=$ one of the equal parts, and $b=$ half the difference between the equal and unequal parts.

Then $a+b=$ the larger part.

And $a-b=$ the less part.

And $(a+b)^2+(a-b)^2=$ the sum of their squares.

But $(a^2+2ab+b^2)+(a^2-2ab+b^2)=2a^2+2b^2$;

And $2a^2+2b^2=$ to twice the square of half the quantity, plus twice the square of half the difference between the equal and unequal parts; that is, the difference between half the quantity and the larger of the unequal parts.

VERIFICATION.

Let $10=7+3$; $7-3=4$; $4\div2=2$; $10\div2=5$.

Then $(5+2)^2+(5-2)^2=2\times5^2+2\times2^2$.

And $49+9=50+8$.

$58=58$.

THEOREM XIV.

171. If any quantity $2a$ be divided into two equal quantities, a and a; and, if any quantity b be added to $2a$, the square of $2a+b$, plus the square of b, will be equal to twice the square of a, plus twice the square of $a+b$.

Now $(2a+b)^2+b^2=4a^2+4ab+b^2+b^2=4a^2+4ab+2b^2$.

But $4a^2+4ab+2b^2=2a^2+2(a+b)^2$;

Therefore $(2a+b)^2+b^2=2a^2+2(a+b)^2$.

VERIFICATION.

Let $a=10$; and $b=4$.

Then $(2\times10+4)^2+4^2=2(10^2)+2(10+4)^2$.

And $576+16=200+392$;

Therefore $592=592$.

SECTION XIII.

INVOLUTION.

ART. **172.** Involution is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, &c. of any given quantity.

173. A *power* is the product of any number or quantity multiplied into itself a certain number of times, and the degree of the power is denoted by an exponent written over the root. Thus a^3 is the third power of a, and a is the root.

174. The *exponent* or *index* shows how many times the root has been used as a factor.

Thus, $a\times a\times a\times a=a^4$, and $x\times x=x^2$.

175. When a quantity is written without any index, it is uniformly considered a unit. Thus, $a=a^1$, and $x=x^1$. Therefore to raise any quantity to any required power, the pupil will see the propriety of the following

RULE. *Multiply the index of the quantity by the index of the power to which it is to be raised, and the result will be the power required.*

Or, multiply the quantity, whether it be simple, fractional, or compound, into itself as many times less one, *as is denoted by the index of the power, and the last product will be the answer.*

176. When the sign of any simple quantity is $+$, all the powers of it will be $+$; and when the sign is $-$, all the *even* powers will be $+$, and the *odd* powers $-$, as is evident from multiplication.

EXAMPLES.

1. What is the fifth power of a? *Ans.* a^5.
2. What is the third power of ax? *Ans.* a^3x^3.

3. Required the square of a^2x. *Ans.* a^4x^2.

4. Required the cube of $-3a^2$. *Ans.* $-27a^6$.

5. Required the fourth power of $-ab^2c^3$. *Ans.* $a^4b^8c^{12}$.

6. Required the square of $-\dfrac{2ax^2}{3b}$. *Ans.* $\dfrac{4a^2x^4}{9b^2}$.

7. Required the fifth power of $2ab^2x^3$. *Ans.* $32a^5b^{10}c^{15}$.

8. Required the sixth power of $\frac{2}{3}a^3x^2$. *Ans.* $\frac{64}{729}a^{18}x^{12}$.

9. Required the third power of $2a^{-2}$. *Ans.* $8a^{-6}$.

10. Required the fourth power of $-3m^{-3}$. *Ans.* $81m^{-12}$.

11. Required the mth power of a^n. *Ans.* a^{mn}.

12. Required the fourth power of $2x^m$. *Ans.* $16x^{4m}$.

13. Required the third power of $\dfrac{3a^2b^3}{4xy^4}$. *Ans.* $\dfrac{27a^6b^9}{64x^3y^{12}}$.

177. Polynomials are involved by multiplying the quantity by itself as many times, *wanting one*, as there are units in the exponent of the power.

14. Let $a+b$ be raised to the fifth power.

$$\begin{array}{rll}
(a+b)^1= & a+b & \text{1st power.} \\
 & \underline{a+b} & \\
 & a^2+ab & \\
 & \underline{\quad +ab+b^2} & \\
(a+b)^2= & a^2+2ab+b^2 & \text{2d power.} \\
 & \underline{a+b} & \\
 & a^3+2a^2b+ab^2 & \\
 & \underline{\quad +a^2b+2ab^2+b^3} & \\
(a+b)^3= & a^3+3a^2b+3ab^2+b^3 & \text{3d power.} \\
 & \underline{a+b} & \\
 & a^4+3a^3b+3a^2b^2+ab^3 & \\
 & \underline{\quad +a^3b+3a^2b^2+3ab^3+b^4} & \\
(a+b)^4= & a^4+4a^3b+6a^2b^2+4ab^3+b^4 & \text{4th power.} \\
 & \underline{a+b} & \\
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 & \\
 & \underline{\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5} & \\
(a+b)^5= & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 & \text{5th power.}
\end{array}$$

Required the third power of $a—b$.

$$\begin{array}{rll}
(a-b)^1= & a-b & \text{1st power.} \\
 & \underline{a-b} & \\
 & a^2-ab & \\
 & \underline{\quad -ab+b^2} & \\
(a-b)^2= & a^2-2ab+b^2 & \text{2d power.} \\
 & \underline{a-b\qquad\qquad} & \\
 & a^3-2a^2b+ab^2 & \\
 & \underline{\quad -a^2b+2ab^2-b} & \\
(a-b)^3= & a^3-3a^2b+3ab^2-b^3 & \text{3d power.}
\end{array}$$

15. Required the fifth power of $x—2y$.

Ans. $x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5$.

16. Required the third power of $a—b+1$.

Ans. $a^3-3a^2b+3a^2+3ab^2-6ab+3a-b^3+3b^2-3b+1$.

17. Required the second power of $2x^2—3x+4$.

Ans. $4x^4-12x^3+25x^2-24x+16$.

18. Required the sixth power of $x—2$.

Ans. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

19. Required the second power of $\frac{2x^2y^{-4}}{3b-4d}$.

Ans. $\frac{4x^4y^{-8}}{9b^2-24bd+16d^2}$.

20. Required the fourth power of $a^m—a^n$.

Ans. $a^{4m}-4a^{3m+n}+6a^{2m+2n}-4a^{m+3n}+a^{4n}$.

21. What is the second power of $2x^2—3x+\frac{1}{2}$? *Ans.*

22. What is the third power of $a+2b—c$? *Ans.*

23. What is the fourth power of $a+b+c+d$? *Ans.*

SECTION XIV.

EVOLUTION, OR THE EXTRACTION OF ROOTS.

Art. **178.** Evolution is the reverse of involution, being the method of finding the roots of any given quantity. It will, therefore, be necessary to trace back the steps of the operation in involution.

Hence, to find any root of a monomial, we adopt the following

Rule. *Extract the required root of the coefficient of the numerical part, and the root of the quantity subjoined for the literal part; and these joined together will be the root required.*

179. If the quantity proposed be a fraction, its root will be found by taking the root both of its numerator and denominator.

180. The square root, the fourth root, or any other even root of an affirmative quantity, may be either plus or minus.

Thus, $\sqrt{a^2}=+a$ or $-a$; and $\sqrt[4]{b^4}=+b$ or $-b$. But the cube root, or any other odd root of a quantity, will have the same sign as the quantity itself. Thus $\sqrt[3]{a^3}=a$; $\sqrt[3]{-a^3}=-a$, and $\sqrt[5]{-a^5}=-a$.

The reason why $+a$ and $-a$ are each the square root of a^2, is obvious, since, by the rule of multiplication, $(+a)\times(+a)$ and $(-a)\times(-a)$ are each equal to a^2.

181. In the case of the cube root, fifth root, &c. of a negative quantity, the rule is equally plain; since, by multiplying, we have $(-a)\times(-a)\times(-a)=-a^3$.

It may here also be stated, that any even root of a negative quantity is unassignable, or, as it is usually called, imaginary.

Thus, $\sqrt{-a^2}$ cannot be determined, as there is no quantity, either positive or negative, that, when multiplied by itself, will produce $-a^2$.

EXAMPLES.

1. Find the square root of $9a^2$.

Here $\sqrt{9a^2}=\sqrt{9}\times\sqrt{a^2}=3\times a=3a$. *Ans*

2. What is the cube root of $8x^3$?

Here $\sqrt[3]{8x^3}=\sqrt[3]{8}\times\sqrt[3]{x^3}=2\times x=2x$. *Ans.*

3. It is required to find the square root of $\frac{a^2b^2}{c^4}$.

Here $\sqrt{\frac{a^2b^2}{c^4}}=\frac{\sqrt{a^2b^2}}{\sqrt{c^4}}=\frac{ab}{c^2}$. *Ans.*

4. What is the cube root of $-\frac{8a^3b^6}{27c^3}$?

Here $-\sqrt[3]{\frac{8a^3b^6}{27c^3}}=-\frac{\sqrt[3]{8}\times\sqrt[3]{a^3b^6}}{\sqrt[3]{27}\times\sqrt[3]{c^3}}=-\frac{2\times ab^3}{3\times c}=-\frac{2ab^3}{3c}$. *Ans.*

5. What is the square root of $16a^4b^8$? *Ans.* $4a^2b^4$.

6. What is the cube root of $-125x^3y^6$? *Ans.* $-5xy^3$.

7. What is the fourth root of $81a^4b^8$? *Ans.* $3ab^2$.

8. What is the fifth root of $\frac{32m^5n^{10}}{243}$? *Ans.* $\frac{2mn^2}{3}$.

9. What is the sixth root of $\frac{729a^6b^{12}}{4096}$? *Ans.* $\frac{3ab^2}{4}$.

NOTE.—Fractions should first be reduced to their lowest terms.

10. Required the square root of $\frac{200a^7}{512a^5}$. *Ans.* $\frac{5a}{8}$.

EVOLUTION OF POLYNOMIALS.

182. To extract the square root.

Since the square of $a+b$ is $a^2+2ab+b^2$; in order to obtain the square root of $a^2+2ab+b^2$, we must consider by what process the quantity $a+b$ can be generally derived from it.

Now, in the first place, we observe that a, the first term of the root, is the square root of a^2, the first term of the square;

and, in addition to this, there still remains $2ab+b^2$, from which b is to be obtained; but $2ab+b^2$ is the same as $(2a+b)b$; and, therefore, b will be determined by dividing the first term of the remainder by *twice* the first term of the root. To complete the operation, *twice* this first term, together with the second, must be multiplied by the second; and, after subtraction, there is no remainder.

183. If the proposed quantity consists of more terms, it is evident that we have only to consider $a+b$ in the place of a, and then by the same process another term of the root will be obtained, and so on; and hence we have the following

GENERAL RULE. *Arrange the terms in the order of the magnitudes of the indices of some one quantity.*

Find the square root of the first term, and subtract its square from the proposed quantity.

Bring down the next two terms, and find the next term of the root by dividing this last quantity by twice the first, and affix it with the proper sign to the divisor.

Multiply this result by the second term of the root, and bring down to the remainder as many terms as make the number equal to that of the next completed divisor; and thus continue the process till the root, or the requisite approximation to it, be obtained.

EXAMPLES.

1. Find the square root of $x^6-6x^3y^2+9y^4$.

$$\begin{array}{r l}
 & x^6-6x^3y^2+9y^4(x^3-3y^2 \\
 & x^6 \\ \cline{2-2}
2x^3-3y^2) & -6x^3y^2+9y^4 \\
 & -6x^3y^2+9y^4. \\ \cline{2-2}
\end{array}$$

184. If the terms had been arranged in the reverse order, as $9y^4-6x^3y^2+x^6$, the root would have been found by a similar process to be $3y^2-x^3$, which differs in its *sign* from the former. But this is immediately explained by the circumstance, that the square root of a quantity is either positive or negative, agreeably to Art. 180.

2. Find the square root of $4x^4-4x^3-3x^2+2x+1$.

$$\begin{array}{rl}
 & 4x^4-4x^3-3x^2+2x+1(2x^2-x-1. \\
 & \underline{4x^4\qquad\qquad\qquad} \\
4x^2-x) & -4x^3-3x^2 \\
 & \underline{-4x^3+\ x^2} \\
4x^2-2x-1) & -4x^2+2x+1 \\
 & \underline{-4x^2+2x+1}
\end{array}$$

3. Extract the square root of $16\,(a^4+1)-24a(a^2+1)+41a^2$.

Having arranged the terms according to the dimensions of a, we have,

$$\begin{array}{rl}
 & 16a^4-24a^3+41a^2-24a+16(4a^2-3a+4. \\
 & \underline{16a^4\qquad\qquad} \\
8a^2-3a) & -24a^3+41a^2 \\
 & \underline{-24a^3+\ 9a^2} \\
8a^2-6a+4) & 32a^2-24a+16 \\
 & \underline{32a^2-24a+16}
\end{array}$$

4. Required the square root of

$$4a^3-16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}}+20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{3}}.$$

$$\begin{array}{rl}
 & 4a^3-16a^{\frac{9}{4}}x^{\frac{2}{3}}-16a^{\frac{3}{2}}x^{\frac{4}{3}}+20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{3}}. \\
 & \underline{4a^3\qquad\qquad\qquad}\qquad\qquad\qquad\Big(2a^{\frac{3}{2}}-4a^{\frac{3}{4}}x^{\frac{2}{3}}+5y^{\frac{5}{6}}c^{\frac{1}{2}}. \\
4a^{\frac{3}{2}}-4a^{\frac{3}{4}}x^{\frac{2}{3}}) & -16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}} \\
 & \underline{-16a^{\frac{9}{4}}x^{\frac{2}{3}}+16a^{\frac{3}{2}}x^{\frac{4}{3}}} \\
4a^{\frac{3}{2}}-8a^{\frac{3}{4}}x^{\frac{2}{3}}+5y^{\frac{5}{6}}c^{\frac{1}{2}}) & 20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{6}} \\
 & \underline{20a^{\frac{3}{2}}y^{\frac{5}{6}}c^{\frac{1}{2}}-40a^{\frac{3}{4}}x^{\frac{2}{3}}y^{\frac{5}{6}}c^{\frac{1}{2}}+25cy^{\frac{5}{6}}.}
\end{array}$$

5. Extract the square root of a^2+x^2.

$$a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7},\text{ \&c.}\right.$$

$$\underline{a^2\qquad\qquad}$$

$$2a+\frac{x^2}{2a}\Big)\ x^2$$

$$\underline{x^2+\frac{x^4}{4a^2}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big)-\frac{x^4}{4a^2}$$

$$\underline{-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}\Big)\ \frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\underline{\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{128a^8}}$$

$$2a+\frac{x^2}{a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}\Big)-\frac{5x^8}{64a^6}+\frac{x^{10}}{128a^8}-$$

$$\underline{-\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8},\text{ \&c.}}$$

6. What is the square root of $x^4-2x^3+3x^2-2x+1$?

Ans. x^2-x+1.

7. What is the square root of $x^6-2x^5+x^4+2x^3-2x^2+1$?

Ans. x^3-x^2+1.

8. What is the square root of $a^4+4a^3b+10a^2b^2+12ab^3+9b^4$?

Ans. $a^2+2ab+3b^2$.

9. Extract the square root $a^4-2a^3+2a^2-a+\frac{1}{4}$.

Ans. $x^2-x+\frac{1}{2}$.

10. What is the square root of $4a^2x^4-12a^3x^3+13a^4x^2-6a^5x+a^6$?

Ans. $2ax^2-3a^2x+a^3$.

11. What is the square root of $\frac{a^2}{b^2}-\frac{4a}{3c}+\frac{4b^2}{9c^2}$?

Ans. $\frac{a}{b}-\frac{2b}{3c}$.

EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

185. As numbers are not expressed in the same manner as algebraic quantities, it is evident that the same rule for extracting the square root of algebraic quantities will not apply to the extracting the roots of numbers without additional considerations. But, if the foregoing rule be assisted by the "Method of Pointing," it will enable us to extract the square root of numbers.

186. Since the square root of 1 is 1;
the square root of 100 is 10;
the square root of 10000 is 100;
the square root of 1000000 is 1000, &c.,

it is evident that the square root of a number of figures *less* than three must consist of only *one* figure; that of a number more than two figures and less than five, of two figures; that of a number more than four figures and less than seven, of three figures, and so on. Whence it follows, that, if a dot be placed over every *alternate* figure, beginning at the unit's place, the number of such points will be the same as the number of figures in the root.

The same rule may be extended to decimals, by first making the decimal places an *even* number, and then commencing the pointing at the unit's place towards the *right* hand over every alternate figure, as before; and the number of such points will be the same as the number of decimal places in the root.

EXAMPLES.

1. Extract the square root of 273529.

ARITHMETICAL FORM.

$$\begin{array}{r}
2\dot{7}3\dot{5}2\dot{9}(523 \\
\underline{25} \\
102)235 \\
\underline{204} \\
1043)3129 \\
\underline{3129}
\end{array}$$

SYMBOLICAL FORM.

$$\begin{array}{r}
273529(500+20+3. \\
500^2=\underline{250000} \\
2\times500+20=1020)23529 \\
\underline{20400} \\
2\times(500+20)+3=1043(3129 \\
\underline{3129}
\end{array}$$

The pupil will perceive that both these operations are performed by Art. 183.

2. Extract the square root of 45796. *Ans.* 214.
3. Extract the square root of 106929. *Ans.* 327.
4. Extract the square root of 36372961. *Ans.* 6031.
5. Extract the square root of 22071204. *Ans.* 4698.
6. Extract the square root of 33.1776. *Ans.* 5.76.
7. Extract the square root of .9409. *Ans.* .97.
8. Extract the square root of .0029997529. *Ans.* .05477.
9. Extract the square root of .001234. *Ans.* .035128+.
10. Extract the square root of 32176552.863844. *Ans.* 5672.438.

CUBE ROOT.

187. Investigation of a rule for extracting the Cube Root of a compound algebraical quantity.

Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we must have the cube root of the latter quantity$=a+b$; and our object is to determine how it may be deduced from it.

Now the first term a of the root is the cube root of a^3, and the first term of the proposed quantity; hence, taking away a^3, we have $3a^2b+3ab^2+b^3$ left to enable us to find b; but $3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)b$. It is therefore manifest, that b will be obtained by dividing the first term of the remainder by *three times* the square of a; and, to complete the divisor, we must add to $3a^2$ *three* times the product of the two terms, or $3ab$, and also the square of the last b^2. Thus, the second term being found, the repetition of a similar process will evidently lead to the root, whatever number of terms the expression may contain. Hence the following

RULE. *Arrange the terms according to the powers of some letter, and extract the root of the first term, which must be a cube or some power of a cube; place this root in the quotient, subtract its cube from the first term, and there will be no remainder.*

Bring down the three next terms for a dividend, and put three times the square of the root just found in the divisor's place, and see how often this is contained in the first term of the dividend, and the quotient is the next term of the root.

Add three times the product of the two terms of the root, plus the square of the last term to the term already in the divisor's place, and the divisor will be completed.

Multiply the complete divisor by the last term of the root; subtract the product from the dividend, and to the remainder connect the three next terms, and proceed as before.

EXAMPLES.

1. Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

$$
\begin{array}{rl}
 & a^3+3a^2b+3ab^2+b^3(a+b. \\
 & a^3 \\
\hline
3a^2+3ab+b^2) & +3a^2b+3ab^2+b^3 \\
 & +3a^2b+3ab^2+b^3 \\
\hline
\end{array}
$$

2. Extract the cube root of $x^6-3x^5+5x^3-3x-1$.

$$\begin{array}{r}
x^6-3x^5+5x^3-3x-1\,(x^2-x-1. \\
x^6 \qquad\qquad\qquad\qquad\qquad\qquad \\
\hline
3x^4-3x^3+x^2)-3x^5+5x^3-3x \qquad\qquad \\
-3x^5+3x^4-x^3 \qquad\qquad\quad \\
\hline
3x^4-6x^3+3x+1)-3x^4+6x^3-3x-1 \qquad\qquad \\
-3x^4+6x^3-3x-1 \qquad\qquad \\
\hline
\end{array}$$

The first divisor is found thus:

$$3\times x^2\times x^2+3x^2(-x)+(-x)^2=3x^4-3x^3+x^2;$$

And the second thus:

$$3(x^2-x)^2+3(x^2-x)\,(-1)+(-1)^2=3x^4-6x^3+3x+1.$$

3. Extract the square root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$$\begin{array}{r}
x^6-6x^5+15x^4-20x^3+15x^2-6x+1\,(x^2-2x+1. \\
x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
\hline
3x^4-6x^3+4x^2)-6x^5+15x^4-20x^3 \qquad\qquad\qquad\qquad\qquad \\
-6x^5+12x^4-8x^3 \qquad\qquad\qquad\qquad\qquad \\
\hline
3x^4-12x^3+15x^2-6x+1)3x^4-12x^3+15x^2-6x+1 \\
3x^4-12x^3+15x^2-6x+1 \\
\hline
\end{array}$$

4. Extract the cube root of $x^3+9x^2+27x+27$.
Ans. $x+3$.

5. Extract the cube root of $1-6y+12y^2-8y^3$.
Ans. $1-2y$.

6. Extract the cube root of $a^6-6a^5+40a^3-90a-64$.
Ans. a^2-2a-4.

7. Extract the cube root of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$. *Ans.* $a+b+c$.

188. Reasoning analogous to that employed in the preceding Art. will show, that, if a point be placed over every *third* figure, beginning at the *unit's* place, the number of points, thus placed, will be the number of digits in the cube root, and attention to Art. 187 will furnish the following operation.

1. Extract the cube root of 1860867.

$$\begin{array}{rrrl}
 & & \cdot\quad\cdot\quad\cdot & a+b+c \\
 & & 1860867 & (100+20+3=123. \\
a^3 = & & \underline{1000000} & =\text{first subtrahend.} \\
3a^2 = & 30000) & 860867 & =\text{first remainder.} \\
3a^2b = & & 600000 & \\
3ab^2 = & & 120000 & \\
b^3 = & & \underline{8000} & \\
 & & \underline{728000} & =\text{second subtrahend.} \\
3(a+b)^2 = & 43200) & 132867 & =\text{second remainder.} \\
3(a+b)^2c = & & 129600 & \\
3(a+b)c^2 = & & 3240 & \\
c^3 = & & \underline{27} & \\
 & & 132867 & =\text{third subtrahend.}
\end{array}$$

This kind of process is the origin of the Rule given on page 248 of the Author's National Arithmetic, to which the pupil is referred.

2. What is the cube root of 31255875? *Ans.* 315.

3. What is the cube root of 37259704? *Ans.* 334.

4. What is the cube root of 116930169? *Ans.* 489.

5. What is the cube root of 508.169592? *Ans.* 7.98.

6. What is the cube root of .724150792? *Ans.* .898.

189. To extract any root of a compound algebraical quantity.

Since $(a+x)^m=a^m+ma^{m-1}x+$ &c., it is obvious, that when the quantities are properly arranged, and the first term of the root is found, the second term of the mth root will be obtained by dividing the second term of the proposed quantity by ma^{m-1}, or by m times the first term, raised to the $(m-1)$th power.

And, if the root thus found, be raised to the mth power, and the result be subtracted from the quantity proposed, and the process be repeated when necessary, any root of a compound quantity may be determined.

The similarity of the processes employed in this and the preceding articles will be immediately noticed, it being observed in the former, the complete powers of a monomial, binomial, trinomial, &c., are subtracted from the proposed quantity by *one*, *two*, *three*, &c. operations; whereas, in the *latter*, the subtraction of the same quantities is effected at *once*. Hence the following

General Rule. 1. *Arrange the terms so that the highest power shall stand in the first term, and let the next higher occupy the second place.*

2. *Find the root of the first term, and place it in the quotient, and having raised this root to the required power, subtract it from the first term, and then bring down the second term for a dividend.*

3. *Involve the root last found to the next inferior power, and multiply it by the index of the given power for a divisor.*

4. *Divide the dividend by the divisor, and the quotient will be the next term of the root.*

5. *Involve the whole root thus found to the required power, which subtract from the given quantity, and divide the first term of the remainder by the same divisor as before.*

6. *Proceed in this manner for the next term of the root, and so proceed until the work is finished.*

EXAMPLES.

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$\begin{array}{l}
a^4-2a^3x+3a^2x^2-2ax^3+x^4\,(a^2-ax+x^2. \\
a^4 \\
\hline
2a^2)-2a^3x \\
\hline
a^4-2a^3x+a^2x^2 \\
\hline
\qquad 2a^2)\,2a^2x^2 \\
\hline
a^4-2a^3x+3a^2x^2-2ax^3+x^4.
\end{array}$$

2. Required the cube root of $x^6+6x^5-40x^4+96x-64$.

$$\begin{array}{l}
x^6+6x^5-40x^3+96x-64\ (x^2+2x-4. \\
x^6 \\
\hline
3x^4)\ 6x^5 \\
\hline
x^6+6x^5+12x^4+8x^3 \\
\hline
3x^4)\ -12x^4 \\
\hline
x^6+6x^5-40x^3+96x-64.
\end{array}$$

3. Required the fourth root of $16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$.

$$\begin{array}{l}
16x^4-96x^3y+216x^2y^2-216xy^3+81y^4\ (2x-3y. \\
16x^4 \\
\hline
32x^3)\ -96x^3y \\
\hline
16x^4-96x^3y+216x^2y^2-216xy^3+81y^4.
\end{array}$$

4. Required the cube root of $m^6-6m^5+40m^3-96m-64$.
Ans. m^2-2m-4.

5. Required the fifth root of $32x^5-80x^4+80x^3-40x^2+10x-1$.
Ans. $2x-1$.

SECTION XIV.

SURDS OR RADICAL QUANTITIES.

Art. 190. Surds or radical quantities, are such as have no exact roots, being usually expressed by means of the radical sign, or fractional indices; in which latter case, the numerator shows the power the quantity is to be raised to, and the numerator its root.

Thus $\sqrt{3}$ or $3^{\frac{1}{2}}$ denotes the square root of 3. $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$ is the cube root of the square of a; and $a^{\frac{m}{n}}$ or $\sqrt[n]{a^m}$ is the nth root of the mth power of a.

191. The quantity 2, or 3, is an irrational quantity or surd, because no number either whole or fractional can be found, which, when multiplied by itself, will produce either 2 or 3. But their proximate values may be found to any degree of exactness by the common rule for extracting the square root.

PROBLEM I.

192. To reduce a rational quantity to the form of a surd or radical quantity.

RULE. *Raise the quantity to a power corresponding to the index of the surd, to which it is to be reduced, and over this new quantity place the radical sign, or proper index, and it will be the form required.*

EXAMPLES.

1. Let 5 be reduced to the form of a square root.

Here $5 \times 5 = 5^2 = 25$; whence $\sqrt{25}$. *Ans.*

2. Reduce $2x^2$ to the form of the cube root.

Here $(2x^2)^3 = 8x^6$; whence $\sqrt[3]{8x^6}$, or $(8x^6)^3$, or $8x^{\frac{3}{6}}$. *Ans.*

3. Let $-2x$ be reduced to the form of the cube root.

Here $(-2x)^3 = -8x^3$; therefore $\sqrt[3]{-8x^3}$. *Ans.*

4. Let $3a^2$ be reduced to the form of the square root.

Ans. $\sqrt{9a^4}$.

5. Let $\frac{x^2}{3}$ be reduced to the form of the cube root.

Ans. $\sqrt[3]{\frac{x^6}{27}}$.

6. Reduce x^3 to the form of the fifth root.

Ans. $\sqrt[5]{x^{15}}$.

7. Let $\frac{x^2}{x-y}$ be reduced to the form of the fourth root.

Ans.

8. Let $(x-y^2)$ be reduced to the form of the square root.

Ans.

193. If a surd be joined to a rational quantity, it may be reduced to a rational by raising the rational part to the required power, and multiplying it by the surd.

9. Let $5\sqrt{7}$ be reduced to a simple radical form.

$$5\sqrt{7}=\sqrt{5\times5}\times\sqrt{7}=\sqrt{25}\times\sqrt{7}=\sqrt{175}.\ \textit{Ans.}$$

10. Let $3\sqrt{a}$ be reduced to a simple radical form.

$$3\sqrt{a}=\sqrt{3\times3}\times\sqrt{a}=\sqrt{9a}.\ \textit{Ans.}$$

11. Let $3\sqrt[3]{3}$ be reduced to a simple radical form.

$$3\sqrt[3]{3}=\sqrt[3]{3\times3\times3}\times\sqrt[3]{3}=\sqrt[3]{27}\times\sqrt[3]{3}=\sqrt[3]{81}.\ \textit{Ans.}$$

12. Let $\frac{1}{3}\sqrt{a}$ be reduced to a simple radical form. *Ans.* $\sqrt{\frac{a}{9}}$.

13. Let $\frac{1}{2}\sqrt[4]{b^2}$ be reduced to a simple radical form.

Ans. $\sqrt[4]{\frac{b^2}{16}}$.

14. Let $3\sqrt[n]{m}$ be reduced to a simple radical form.

Ans. $\sqrt[n]{3^n m}$.

15. Let $\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}$ be reduced to a simple radical form.

$$\frac{x+1}{x-1}\sqrt{\frac{x-1}{x+1}}=\sqrt{\left(\frac{x+1}{x-1}\right)^2\left(\frac{x-1}{x+1}\right)}=\sqrt{\frac{x^3+x^2-x-1}{x^3-x^2-x+1}}=$$

$$\sqrt{\frac{x+1}{x-1}},\text{ or }\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}.\ \textit{Ans.}$$

16. Let $\frac{2x}{3}\sqrt[3]{\frac{9}{4x^2}}$ be reduced to a simple radical form.

Ans. $\sqrt[3]{\frac{2x}{3}}$.

PROBLEM II.

194. To reduce quantities or numbers of different indices to others that shall have a given index.

RULE. *Divide the indices of the proposed quantities or numbers by the given index, and the quantities will be the new indices for those quantities.*

Then, over the said quantities with their new indices, place the given index, and they will be the equivalent quantities required.

EXAMPLES.

1. Reduce $4^{\frac{1}{2}}$ and $8^{\frac{1}{3}}$ to other quantities of the same value, each having the common index $\frac{1}{6}$.

Here $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=\frac{6}{2}=3$, the first index,

And $\frac{1}{3}\div\frac{1}{6}=\frac{1}{3}\times\frac{6}{1}=\frac{6}{3}=2$, the second index.

Whence $(4^3)^{\frac{1}{6}}=4^{\frac{1}{2}}$; and $(8^2)^{\frac{1}{6}}=8^{\frac{1}{3}}$. *Ans.*

195. The truth of this rule will be evident, if 4 be raised to the 3d power, and the 6th root extracted, and that root shall be equal to the square root of 4.

Thus $4\times4\times4=64$; $\sqrt[6]{64}=2$; $\sqrt{4}=2$.

And, if 8 be raised to the 2d power, and the 6th root extracted, the result will be equal to the cube root of 8.

Thus, $8\times8=64$; $\sqrt[6]{64}=2$; $\sqrt[3]{\ }=2$.

2. Reduce $3^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ to the common index $\frac{1}{6}$.

Ans. $\sqrt[\frac{1}{6}]{3^2}=\sqrt[\frac{1}{6}]{9}$; $\sqrt[\frac{1}{6}]{5^3}=\sqrt[\frac{1}{6}]{125}$.

3. Reduce a^2 and $a^{\frac{1}{2}}$ to quantities that shall have the common index $\frac{1}{4}$. *Ans.* $\sqrt[4]{a^8}$ and $\sqrt[4]{a^2}$.

4. Reduce $3a^{\frac{1}{2}}$ and $2a^{\frac{3}{4}}$ to the quantities that shall have the common index $\frac{1}{8}$. *Ans.* $\sqrt[8]{3a^4}$ and $\sqrt[8]{2a^6}$.

5. Reduce $5x^{\frac{1}{4}}$ and $6y^{\frac{1}{3}}$ to quantities having the common index $\frac{1}{12}$. *Ans.* $\sqrt[12]{5x^3}$ and $\sqrt[12]{6y^4}$.

6. Reduce $a^{\frac{m}{n}}$ and $b^{\frac{p}{s}}$ to quantities having a common index.

$$a^{\frac{m}{n}}=a^{\frac{m}{n}\times\frac{s}{s}}=a^{\frac{ms}{ns}}\text{; and }b^{\frac{p}{s}}=b^{\frac{p}{s}\times\frac{n}{n}}=b^{\frac{np}{ns}}.$$

Therefore $a^{\frac{m}{n}}=(a^{ms})^{\frac{1}{ns}}$; and $b^{\frac{p}{s}}=(b^{np})^{\frac{1}{ns}}$.

PROBLEM III.

To reduce surds to a common index.

RULE. *Reduce the indices of the quantities to a common denominator, and then involve each quantity to the power denoted by its numerator.*

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to quantities having a common index.

We first reduce the fractional indices, $\frac{1}{2}$ and $\frac{1}{3}$, to a common denominator, and find them to be $\frac{3}{6}$ and $\frac{2}{6}$, which have the same value as $\frac{1}{2}$ and $\frac{1}{3}$.

Hence $3^{\frac{1}{2}}=3^{\frac{3}{6}}=(3^3)^{\frac{1}{6}}=27^{\frac{1}{6}}$ or $\sqrt[6]{27}$.

And $4^{\frac{1}{3}}=4^{\frac{2}{6}}=(4^2)^{\frac{1}{6}}=16^{\frac{1}{6}}$ or $\sqrt[6]{16}$.

2. Reduce $4^{\frac{1}{3}}$ and $6^{\frac{1}{4}}$ to equal quantities, that shall have the same index.

$$\tfrac{1}{3}\text{ and }\tfrac{1}{4}=\tfrac{4}{12}\text{ and }\tfrac{3}{12}.$$

Therefore $4^{\frac{1}{3}}=4^{\frac{4}{12}}=(4^4)^{\frac{1}{12}}=(256)^{\frac{1}{12}}$ or $\sqrt[12]{256}$. *Ans.*

And $6^{\frac{1}{4}}=6^{\frac{3}{12}}=(6^3)^{\frac{1}{12}}=(216)^{\frac{1}{12}}$ or $\sqrt[12]{216}$. *Ans.*

3. Reduce $2^{\frac{2}{3}}$ and $3^{\frac{1}{2}}$ to equal quantities having a common index. *Ans.* $\sqrt[6]{16}$ and $\sqrt[6]{27}$.

4. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{4}}$ to equal quantities having a common index. *Ans.* $\sqrt[4]{a^2}$ and $\sqrt[4]{b}$.

5. Reduce $x^{\frac{1}{m}}$ and $y^{\frac{1}{n}}$ to quantities having a common index. *Ans.* $\sqrt[mn]{x^n}$ and $\sqrt[mn]{y^m}$.

PROBLEM IV.

196. To reduce surds to their most simple form.

RULE. *Resolve the given number or quantity into two factors, one of which shall be the greatest power contained in it, and set the root of this power before the remaining part, with the proper radical sign between them.*

NOTE.—When the given surd contains no factor, which is an exact power, it is already in its most simple form. Thus $\sqrt{15}$ cannot be reduced lower, because neither of the factors 5 or 3 is a square.

EXAMPLES.

1. Let $\sqrt{48}$ be reduced to its most simple form.

We divide 48 into two factors, 16 and 3, 16 being the greatest power of the required root. We therefore extract the square root of 16, and write its root, 4, before the other factor, having the sign prefixed to the surd.

Thus $\sqrt{48}=\sqrt{16\times3}=4\sqrt{3}$. *Ans.*

2. Let $\sqrt[3]{108}$ be reduced to its most simple form.

In this question, we find the factors of 108 to be 27 and 4; 27 being the largest possible factor of which the cube root could be extracted. The operation, therefore, is

Thus $\sqrt[3]{108}=\sqrt[3]{27\times4}=3\sqrt[3]{4}$. *Ans.*

3. Let $\sqrt{75}$ be reduced to its most simple form.

Ans. $5\sqrt{3}$.

4. Let $\sqrt[4]{80}$ be reduced to its most simple form.

Ans. $2\sqrt[4]{5}$.

5. Reduce $\sqrt{27a^3x^5}$ to its simplest form.

Here $\sqrt{27a^3x^5}=\sqrt{9a^2x^4\times3ax}=\sqrt{9a^2x^4}\times\sqrt{3ax}=3ax^2\sqrt{3ax}$.

6. Reduce $\sqrt[3]{54a^5x^4}$ to its simplest form.

Ans. $3ax\sqrt[3]{2a^2x}$.

PROBLEM V.

197. When any number or quantity is prefixed to the surd, that quantity must be multiplied by the root of the factor, as in Art. 196, and the product must then be joined to the other part as before.

EXAMPLES.

1. Let $2\sqrt{32}$ be reduced to its most simple form.

Here $2\sqrt{32}=2\sqrt{16\times2}=2\times4\sqrt{2}=8\sqrt{2}$. *Ans.*

In performing this question, we first find the factors of 32, which are 16 and 2.

We then extract the square root of 16, and multiply its root, 4, by the number prefixed to the surd, and find the product to be 8, to which we subjoin the surd 2.

198. This and all similar questions might have been performed by squaring the number prefixed to the surd, and then multiplying this number by the surd. Let this product be divided into two factors, as before; and the square of the former prefixed to the latter will give the answer.

Thus $2\sqrt{32}=\sqrt{2\times2\times32}=\sqrt{128}=\sqrt{64\times2}=8\sqrt{2}$. *Ans.*

2. Let $5\sqrt[3]{24}$ be reduced to its most simple form.

Here $5\sqrt[3]{24}=5\sqrt[3]{8\times3}=5\times2\sqrt[3]{3}=10\sqrt[3]{3}$,

Or $5\sqrt[3]{24}=\sqrt[3]{5\times5\times5\times24}=\sqrt[3]{3000}=\sqrt[3]{1000\times3}=10\sqrt[3]{3}$.

3. Reduce $2\sqrt[3]{40}$ to simple terms. *Ans.* $4\sqrt[3]{5}$.

PROBLEM VI.

199. A fractional surd may be reduced to a more convenient form by multiplying both the numerator and denominator by such a number or quantity as will make the denominator a complete power of the kind required, and then joining its root to the part of the question.

EXAMPLES.

1. Let $\sqrt{\frac{2}{5}}$ be reduced to its most simple form.

$$\sqrt{\tfrac{2}{5}\times\tfrac{5}{5}}=\sqrt{\tfrac{10}{25}}=\sqrt{\tfrac{1}{25}\times\tfrac{10}{1}}=\tfrac{1}{5}\sqrt{10}.\quad \textit{Ans.}$$

2. Let $\sqrt[3]{\frac{2}{3}}$ be reduced to its most simple form.

$$\sqrt[3]{\tfrac{2}{3}}=\sqrt[3]{\tfrac{2}{3}\times\tfrac{9}{9}}=\sqrt[3]{\tfrac{18}{27}}=\sqrt[3]{\tfrac{1}{27}\times\tfrac{18}{1}}=\tfrac{1}{3}\sqrt[3]{18}.\quad \textit{Ans.}$$

3. Let $\sqrt{\frac{2}{7}}$ be reduced to its most simple form.
Ans. $\frac{1}{7}\sqrt{14}$.

4. Let $\sqrt[3]{\frac{2}{5}}$ be reduced to its most simple form.
Ans. $\frac{1}{5}\sqrt[3]{50}$.

5. Let $\sqrt[4]{\frac{1}{2}}$ be reduced to its most simple form.
Ans. $\frac{1}{2}\sqrt[4]{8}$.

EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. What is the most simple form of $\sqrt{125}$? *Ans.* $5\sqrt{5}$.

2. What is the most simple form of $\sqrt{80a^2x^3}$?
Ans. $4ax\sqrt{5x}$.

3. What is the most simple form of $\sqrt[3]{189a^4b^3c^2}$?
Ans. $3ab\sqrt[3]{7ac^2}$.

4. What is the most simple form of $7\sqrt{80}$?
Ans. $28\sqrt{5}$.

5. What is the most simple form of $\frac{3}{7}\sqrt{\frac{4}{5}}$? *Ans.* $\frac{6}{7}\sqrt{\frac{1}{5}}$.

6. What is the most simple form of $\frac{3}{11}\sqrt{\frac{4}{7}}$? *Ans.* $\frac{6}{11}\sqrt{\frac{1}{7}}$.

7. Let $\sqrt{96a^2x^3}$ be reduced to its most simple form.
Ans. $4ax\sqrt{6x}$.

8. Let $\frac{3}{7}\sqrt[3]{56x^4+64y^3}$ be reduced to its most simple form.
Ans. $\frac{6}{7}\sqrt[3]{(7x^4+8y^3.)}$

PROBLEM VII.

200. To add surd quantities together.

I. When the radicals are similar, annex the radical part to the sum of the coefficients.

EXAMPLES.

1. Add $7\sqrt{2}$ to $5\sqrt{2}$. *Ans.* $12\sqrt{2}$.

2. Add $5\sqrt{ab}$ to $3\sqrt{ab}$. *Ans.* $8\sqrt{ab}$.

3. Add $a\sqrt{xy}$ to $b\sqrt{x}$. *Ans.* $(a+b)\sqrt{xy}$.

4. Add $7\sqrt{a^2-y}$ to $y\sqrt{a^2-y}$. *Ans.* $(7+y)\sqrt{a^2-y}$.

II. When the radical parts are dissimilar, they must, if possible, be first made similar by Art. 197, and proceed as above.

But, if the surd be not the same in all the quantities, they can only be added by the signs $+$ and $-$.

5. Add $\sqrt{18}$ and $\sqrt{32}$ together.

First $\sqrt{18}=\sqrt{9\times2}=3\sqrt{2}$,

And $\sqrt{32}=\sqrt{16\times2}=4\sqrt{2}$,

Then $3\sqrt{2}+4\sqrt{2}=7\sqrt{2}$. *Ans.*

6. Required the sum of $\sqrt[3]{375}$ and $\sqrt[3]{192}$.

First $\sqrt[3]{375}=\sqrt[3]{125\times3}=5\sqrt[3]{3}$,

And $\sqrt[3]{192}=\sqrt[3]{64\times3}=4\sqrt[3]{3}$,

Then $5\sqrt[3]{3}+4\sqrt[3]{3}=9\sqrt{3}$. *Ans.*

7. Required the sum of $\sqrt{27}$ and $\sqrt{48}$. *Ans.* $7\sqrt{3}$.

8. Required the sum of $\sqrt{50}$ and $\sqrt{72}$. *Ans.* $11\sqrt{2}$.

9. Find the sum of $\sqrt{180}$ and $\sqrt{405}$. *Ans.* $15\sqrt{5}$.

10. It is required to find the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$.
Ans. $5\sqrt[3]{5}$.

11. Find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$. *Ans.* $32\sqrt[3]{2}$.

12. Find the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$. *Ans.* $\frac{3}{4}\sqrt[3]{2}$.

13. Required the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$.
Ans. $(3a+20a^2)\sqrt{b}$.

PROBLEM VIII.

201. To find the difference of surd quantities.

RULE. *Prepare the quantities as in addition of surds, then subtract the coefficients, and, to the remainder, annex the common surd for the difference of the surds required.*

But, if the quantities have no common surd, they can only be subtracted by means of the sign —.

EXAMPLES.

1. From $\sqrt{320}$ take $\sqrt{80}$.

First $\sqrt{320}=\sqrt{64\times5}=8\sqrt{5}$,

And $\sqrt{80}=\sqrt{16\times5}=4\sqrt{5}$,

Then $8\sqrt{5}-4\sqrt{5}=4\sqrt{5}$. *Ans.*

2. Find the difference between $\sqrt[3]{128}$ and $\sqrt[3]{54}$.

First $\sqrt[3]{128}=\sqrt[3]{64\times2}=4\sqrt[3]{2}$,

And $\sqrt[3]{54}=\sqrt[3]{27\times2}=3\sqrt[3]{2}$,

Then $4\sqrt[3]{2}-3\sqrt[3]{2}=\sqrt[3]{2}$. *Ans.*

3. Required the difference between $2\sqrt{50}$ and $\sqrt{18}$.
Ans. $7\sqrt{2}$.

4. What is the difference between $2\sqrt[3]{320}$ and $3\sqrt[3]{40}$?
Ans. $2\sqrt[3]{5}$.

5. Required the difference of $\sqrt{75}$ and $\sqrt{48}$. *Ans.* $\sqrt{3}$.

6. Required the difference of $\sqrt[3]{256}$ and $\sqrt[3]{32}$.
Ans. $2\sqrt[3]{4}$.

7. Required the difference of $\sqrt[3]{\frac{3}{4}}$ and $\sqrt[3]{\frac{2}{9}}$.
Ans. $\frac{1}{6}\sqrt[3]{6}$.

8. Required the difference of $\sqrt[3]{\frac{3}{5}}$ and $\sqrt[3]{\frac{25}{9}}$.
Ans. $\frac{2}{15}\sqrt[3]{75}$.

9. Find the difference of $\frac{3}{7}\sqrt[3]{a^3b}$ and $\frac{2}{5}\sqrt[3]{a^6b}$.
Ans. $\left(\frac{15a}{35}-\frac{14a}{35}\right)\sqrt{b}$.

10. From $\sqrt{4ax^2}$ take $3x\sqrt{9a}$. *Ans.* $-7x\sqrt{a}$.

PROBLEM IX.

202. To multiply surd quantities together.

RULE. *When the surds are of the same kind, find the product of the rational parts, and the product of the surds, and the two, joined together with the common radical signs between them; will give the whole product required, which may be reduced to its most simple form by* Art. 199.

203. If the surds are of different kinds, they must be reduced to a common index, and then multiplied together as before.

204. Powers and roots of the same quantity are multiplied by adding their exponents.

EXAMPLES.

1. Find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here $3\sqrt{8}$
Multiplied by $2\sqrt{6}$
Gives $6\sqrt{48}=6\sqrt{(16\times3)}=24\sqrt{3}$. *Ans.*

2. Find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.

Here $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$
Multiplied by $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$
Gives $\frac{3}{8}\sqrt[3]{\frac{5}{9}}=\frac{3}{8}\sqrt[3]{(\frac{5}{9}\times\frac{3}{3})}=\frac{3}{8}\sqrt[3]{(\frac{15}{27})}=$
$\frac{3}{8}\sqrt[3]{(\frac{1}{27}\times\frac{15}{1})}=(\frac{3}{8}\times\frac{1}{3})\sqrt[3]{15}=\frac{1}{8}\sqrt[3]{15}$. *Ans.*

3. Multiply $2^{\frac{1}{2}}$ by $3^{\frac{1}{3}}$.

Here $2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}$,

And $3^{\frac{1}{3}}=3^{\frac{2}{6}}=(3^2)^{\frac{1}{6}}=9^{\frac{1}{6}}$

$72^{\frac{1}{6}}$. *Ans.*

4. Multiply $5\sqrt{a}$ by $3\sqrt[3]{a}$.

Here $5\sqrt{a}=5a^{\frac{1}{2}}=5a^{\frac{3}{6}}$,

And $3\sqrt[3]{a}=3a^{\frac{1}{3}}=3a^{\frac{2}{6}}$

$15a^{\frac{5}{6}}=15\sqrt[6]{a^5}$. *Ans.*

5. Multiply $4\sqrt{12}$ by $3\sqrt{2}$. *Ans.* $24\sqrt{6}$.

6. Multiply $3\sqrt{2}$ by $2\sqrt{8}$. *Ans.* 24.

7. Multiply $\frac{1}{3}\sqrt[3]{4}$ by $\frac{3}{4}\sqrt[3]{12}$. *Ans.* $\frac{1}{2}\sqrt[3]{6}$.

8. Multiply $\frac{5}{3}\sqrt{\frac{3}{8}}$ by $\frac{9}{10}\sqrt{\frac{3}{5}}$. *Ans.* $\frac{9}{4}\sqrt{\frac{1}{10}}$.

9. Multiply $7\sqrt[3]{18}$ by $5\sqrt[3]{4}$. *Ans.* $70\sqrt[3]{9}$.

10. Multiply $\frac{1}{4}\sqrt[3]{6}$ by $\frac{2}{15}\sqrt[3]{17}$. *Ans.* $\frac{1}{30}\sqrt[3]{102}$.

11. Multiply $2a^{\frac{2}{3}}$ by $a^{\frac{4}{3}}$. *Ans.* $2a^2$.

12. Multiply $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$. *Ans.* $\sqrt[12]{(a+b)^{13}}$.

13. Multiply $x-\sqrt{xy}+y$ by $\sqrt{x}+\sqrt{y}$.

By expressing the surds with fractional indices, we have,

$$\begin{array}{l}
x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y \\
\underline{x^{\frac{1}{2}}+y^{\frac{1}{2}}\qquad\qquad} \\
x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y \\
\underline{\qquad +xy^{\frac{1}{2}}-x^{\frac{1}{2}}y+y^{\frac{3}{2}}} \\
x^{\frac{3}{2}}\qquad\qquad\qquad\qquad +y^{\frac{3}{2}}.
\end{array}$$

14. Multiply $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

$$\begin{array}{l}
a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}} \\
\underline{a^{\frac{1}{2}}-b^{\frac{1}{3}}\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
a^3+a^{\frac{5}{2}}b^{\frac{1}{3}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b+ab^{\frac{4}{3}}+a^{\frac{1}{2}}b^{\frac{5}{3}} \\
\underline{\qquad -a^{\frac{5}{2}}b^{\frac{1}{3}}-a^2b^{\frac{2}{3}}-a^{\frac{3}{2}}b-ab^{\frac{4}{3}}-a^{\frac{1}{2}}b^{\frac{5}{3}}-b^2} \\
a^3\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -b^2.\ \textit{Ans.}
\end{array}$$

15. Multiply $\sqrt{a}+\sqrt{b}+\sqrt{c}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.
Ans. $a+b-c+2\sqrt{ab}$.

PROBLEM X.

205. To divide one surd quantity by another.

Rule. *When the surds are of the same kind, find the quotient of the rational parts, and the quotients of the surds, and the two joined together, with the common radical sign between them, will give the whole quotient required.*

But, if the surds are of different kinds, they must be reduced to a common index and be divided as above.

It is also to be observed, that the quotients of different powers or roots of the same quantity are found by subtracting their indices.

EXAMPLES.

1. Divide $6\sqrt{96}$ by $3\sqrt{8}$.

Here $\frac{6\sqrt{96}}{3\sqrt{8}}=2\sqrt{12}=2\sqrt{4\times3}=(2\times2)\sqrt{3}=4\sqrt{3}$. *Ans.*

2. Divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\frac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}=4\sqrt{9\times2}=(4\times3)\sqrt{2}=12\sqrt{2}$. *Ans.*

3. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\frac{8\sqrt[3]{512}}{4\sqrt[3]{2}}=2\sqrt[3]{256}=2\sqrt[3]{64\times4}=8\sqrt[3]{4}$. *Ans.*

4. Divide 12 times the cube root of 280 by 3 times the cube root of 5.

Here $\frac{12\sqrt[3]{280}}{3\sqrt[3]{5}}=4\sqrt[3]{56}=4\sqrt[3]{8\times7}=8\sqrt[3]{7}$. *Ans.*

5. Divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

6. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. *Ans.* $2\sqrt[3]{4}$.

7. Divide $4\sqrt{50}$ by $2\sqrt{5}$. *Ans.* $2\sqrt{10}$.

8. Divide $6\sqrt[3]{100}$ by $3\sqrt[3]{5}$. *Ans.* $2\sqrt[3]{20}$.

9. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$. *Ans.*

10. Divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{b}$. *Ans.*

206. Since the division of surds is performed by subtracting their indices, it is evident, that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator by changing the sign of its index or exponent.

EXAMPLES.

1. Let $\frac{1}{a}$ be expressed by a negative index.

$$\frac{1}{a}=\frac{a^{-1}}{1}=a^{-1}.$$

2. Let $\frac{1}{a^n}$ be expressed by a negative index.

$$\frac{1}{a^n}=\frac{a^{-n}}{1}=a^{-n}.$$

3. Let $\frac{b}{a^2}$ be expressed by a negative index.

$$\frac{b}{a^2}=\frac{ba^{-2}}{1}=ba^{-2}.$$

4. Let $\frac{1}{a^2}$ be expressed by a negative index.

$$\frac{1}{a^2}=\frac{a^{-2}}{1}=a^{-2}.$$

5. Let $a^{-\frac{1}{2}}$ be expressed by a positive index.

$$a^{-\frac{1}{2}}=\frac{a^{-\frac{1}{2}}}{1}=\frac{1}{a^{\frac{1}{2}}}.$$

6. Let $\sqrt{\frac{1}{a+x}}$ be expressed by a negative index.

Ans. $(a+x)^{-2}$.

7. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed by a positive index.

Ans. $\frac{1}{a(a^2-x^2)^{\frac{1}{3}}}$.

8. What is the value of $\frac{a^m}{a^m}$?

$$\frac{a^m}{a^m} = a^{m-m} = a^0 = 1.$$

Whence it follows, that a^0 is a symbol equivalent to unity, and may consequently be always replaced by 1, whenever it occurs. This, however, has been demonstrated in a previous article.

PROBLEM XI.

207. To involve or raise surd quantities to any power.

Let $a^{\frac{h}{g}}$ represent a surd quantity, then by Art. 204 its square will be

$$a^{\frac{h}{g}} \times a^{\frac{h}{g}} = a^{\frac{h+h}{g}} = a^{\frac{2h}{g}}.$$

Therefore, to involve a surd to any required power, we adopt the following

RULE. *When the surd is a simple quantity, multiply its index by 2 for the square, 3 for the cube, &c., and it will give the power of the surd part, which being annexed to the proper power of the rational parts, will give the whole power required.*

If the surd be a compound quantity, multiply it by itself the requisite number of times.

EXAMPLES.

1. What is the square of $3a^{\frac{1}{3}}$?

$$3a^{\frac{1}{3}} \times^{\frac{2}{1}} = 3a^{\frac{2}{3}} = 9\sqrt[3]{a^2}. \quad \textit{Ans.}$$

2. What is the cube of $\frac{2}{3}\sqrt{3}$?

Here $(\frac{2}{3}\sqrt{3})^3 = \frac{8}{27}\sqrt{27} = \frac{8}{27}\sqrt{(9 \times 3)} = \frac{8}{9}\sqrt{3}$. *Ans.*

3. Required the square of $3\sqrt[3]{3}$. *Ans.* $9\sqrt[3]{9}$.

4. Required the cube of $17\sqrt{21}$. *Ans.* $103173\sqrt{21}$.

5. What is the fourth power of $\frac{1}{6}\sqrt{6}$? *Ans.* $\frac{1}{36}$.

6. Required the cube of $\sqrt{3}$. *Ans.* $3\sqrt{3}$.

7. Required the third power of $\frac{1}{3}\sqrt{3}$. *Ans.* $\frac{1}{9}\sqrt{3}$.

8. Required the fourth power of $\frac{1}{2}\sqrt{2}$. *Ans.* $\frac{1}{4}$.

9. What is the mth power of $a^{\frac{1}{n}}$? *Ans.* $a^{\frac{m}{n}}$.

10. Required the square of $2+\sqrt{3}$. *Ans.* $7+4\sqrt{3}$.

11. What is the $\frac{r}{s}$-th power of $a^{\frac{p}{q}}$? *Ans.* $a^{\frac{pr}{qs}}$.

PROBLEM XII.

208. To find the roots of surd quantities.

RULE. *When the surd is a simple quantity, multiply its index by* $\frac{1}{2}$ *for the square root, by* $\frac{1}{3}$ *for the cube root, &c., and it will give the root for the surd part, which being annexed to the root of the rational part, will give the whole root required.*

The truth of this rule may be illustrated by the following

EXAMPLES.

1. What is the cube root of the square root of 64?

The square root of $64=\sqrt{64}=64^{\frac{1}{2}}=8$.

And the cube root of $8=\sqrt[3]{8}=8^{\frac{1}{3}}=2$. *Ans.*

209. The same result would have been obtained, if we had multiplied the index, $(\frac{1}{2})$, of the given quantity by the index of the required root, $(\frac{1}{3})$. The product of which is $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$, and if we had considered this $(\frac{1}{6})$ the index of the root to be extracted of the given quantity 64, the operation would have been thus; $\sqrt[\frac{1}{6}]{64}=2$, *Ans.* as before.

2. Required the cube root of the square root of a.

$a^{\frac{1}{2}\times\frac{1}{3}}=a^{\frac{1}{6}}$. *Ans.*

3. Required the fourth root of $\sqrt{3}$. $3^{\frac{1}{2}\times\frac{1}{4}}=3^{\frac{1}{8}}$. *Ans.*

4. What is the square root of $9\sqrt[3]{3}$?

Here $(9\sqrt[3]{3})^{\frac{1}{2}}=9^{\frac{1}{2}}\times3^{\frac{1}{3}\times\frac{1}{2}}=9^{\frac{1}{2}}\times3^{\frac{1}{6}}=3\sqrt[6]{3}$.

5. What is the square root of 10^3?

$10^3=1000:\sqrt{1000}=10\sqrt{10}$. *Ans.*

6. What is the cube root of $\frac{27}{64}\sqrt{a}$? *Ans.* $\frac{3}{4}\sqrt[\frac{1}{6}]{a}$.

7. What is the square root of $\frac{16}{25}a^5$? *Ans.* $\frac{4}{5}a^2\sqrt{a}$.

PROBLEM XIII.

210. To find factors or multipliers, that shall cause any surds to become rational.

I. When the surd is a monomial, multiplying it by the same quantity with such an index, as when added to the index of the given quantity will make it a unit, will cause the surd to become rational.

The quantity $\sqrt{a}$ or $a^{\frac{1}{2}}$ is made rational by multiplying it by $\sqrt{a}$ or $a^{\frac{1}{2}}$.

Thus, $\sqrt{a}\times\sqrt{a}$, or $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$.

And it will be rational, if $a^{\frac{1}{3}}$ be multiplied by $a^{\frac{2}{3}}$, thus, $a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a$.

Also, if $a^{\frac{4}{5}}$ be multiplied by $a^{\frac{1}{5}}$, it will be rational; thus, $a^{\frac{4}{5}}\times a^{\frac{1}{5}}=a$.

EXAMPLES.

1. What factor will make $x^{\frac{2}{3}}$ rational? *Ans.* $x^{\frac{1}{3}}$.

2. What factor will make $y^{\frac{2}{7}}$ rational? *Ans.* $y^{\frac{5}{7}}$.

3. What factor will cause a^{-3} to become rational?

Ans. a^4.

II. When the surd is a binomial or residual quantity, and both the terms are even roots, to find a factor.

In Art. 159, we have shown, that the product of the sum and difference of any two quantites is equal to the difference of their squares; therefore, when one or both of the terms are even roots, we multiply the given binomial or residual by the same expression or quantity with the sign of one of its terms changed.

NOTE. — It is sometimes necessary to repeat the operation.

EXAMPLES.

1. To find a multiplier or factor, that shall make $4+\sqrt{5}$ rational.

$$\begin{array}{ll} \text{Given surd,} & 4+\sqrt{5} \\ \text{Multiplier,} & \underline{4-\sqrt{5}} \\ & 16+4\sqrt{5} \\ & \underline{\quad -4\sqrt{5}-5} \\ \text{Product,} & 16 \qquad -5=11 \text{ rational quantity.} \end{array}$$

2. Find a factor, that shall make $\sqrt{a}+\sqrt{b}$ rational.

$$\begin{array}{l} \sqrt{a}+\sqrt{b} \\ \underline{\sqrt{a}-\sqrt{b}} \\ a+\sqrt{ab} \\ \underline{\quad -\sqrt{ab}-b} \\ a \qquad -b \text{ rational quantity.} \end{array}$$

3 What factor will make $1+\sqrt{3}$ rational?

$$\begin{array}{l} 1+\sqrt{3} \\ \underline{1-\sqrt{3}} \\ 1+\sqrt{3} \\ \underline{\quad -\sqrt{3}-3} \\ 1 \qquad -3=-2 \text{ rational quantity.} \end{array}$$

4. What factor will make $\sqrt{5}-\sqrt{1}$ rational?

$$\begin{array}{l} \sqrt{5}-\sqrt{1} \\ \underline{\sqrt{5}+\sqrt{1}} \\ 5-\sqrt{5} \\ \underline{\quad +\sqrt{5}-1} \\ 5 \qquad -1=4 \text{ rational quantity.} \end{array}$$

5. Find multipliers, that shall make $\sqrt[4]{5}+\sqrt[4]{3}$ rational.

$$\begin{array}{lr}
\sqrt[4]{5}+\sqrt[4]{3} & \\
\sqrt[4]{5}-\sqrt[4]{3} & \\
\hline
\sqrt{5}+\sqrt[4]{15} & \\
\quad -\sqrt[4]{15} & -\sqrt{3} \\
\hline
\sqrt{5} & -\sqrt{3} \\
\sqrt{5} & +\sqrt{3} \\
\hline
5-\sqrt{}-\sqrt{15} & \\
\quad +\sqrt{}+\sqrt{15} & -3 \\
\hline
5 & -3=2
\end{array}$$

rational quantity.

6. What multipliers will make $\sqrt{5}-\sqrt{x}$ rational?

$$\begin{array}{lr}
\sqrt{5}-\sqrt{x} & \\
\sqrt{5}+\sqrt{x} & \\
\hline
5-\sqrt{5x} & \\
\quad +\sqrt{5x} & -x \\
\hline
5 & -x
\end{array}$$

rational quantity.

III. A trinomial surd may be rendered rational by changing the sign of one of its terms for the multiplier.

EXAMPLES.

1. To find multipliers, that shall make $\sqrt{7}+\sqrt{3}-\sqrt{2}$ rational.

$$\begin{array}{l}
\sqrt{7}+\sqrt{3}-\sqrt{2} \\
\sqrt{7}+\sqrt{3}+\sqrt{2} \\
\hline
7+\sqrt{21}-\sqrt{14} \\
\quad +\sqrt{21}+3-\sqrt{6} \\
\qquad\qquad +\sqrt{14}+\sqrt{6}-2 \\
\hline
8+2\sqrt{21} \\
-8+2\sqrt{21} \\
\hline
-64-16\sqrt{21} \\
\quad +16\sqrt{21}+84 \\
\hline
84-64=20
\end{array}$$

rational quantity.

2. Find a factor, that will make $\sqrt{8}-\sqrt{1}-\sqrt{3}$ rational.

$$\begin{array}{l}
\sqrt{8}-\sqrt{1}-\sqrt{3}\\
\underline{\sqrt{8}+\sqrt{1}+\sqrt{3}}\\
\quad 8-\sqrt{8}-\sqrt{24}\\
\qquad +\sqrt{8}-1-\sqrt{3}\\
\qquad\qquad \underline{+\sqrt{24}-\sqrt{3}-3}\\
\quad 4-2\sqrt{3}\\
\quad \underline{4+2\sqrt{3}}\\
16-8\sqrt{3}\\
\quad \underline{+8\sqrt{3}-12}\\
16-12=4 \text{ rational quantity.}
\end{array}$$

QUESTIONS FOR EXERCISE.

1. Find a multiplier, that shall make $\sqrt{5}-\sqrt{2}$ rational.
Ans.

2. Find a multiplier, that shall make $\sqrt{7}+\sqrt{6}$ rational.
Ans.

3. Find a multiplier, that shall make $\sqrt{10}-\sqrt{2}$ rational.
Ans.

4. Find a multiplier, that shall make $\sqrt{a}+\sqrt{b}+\sqrt{c}$ rational.
Ans.

. Find a multiplier, that shall make $\sqrt[4]{3}-\sqrt[4]{1}$ rational.
Ans.

PROBLEM XIV.

ART. **211.** To reduce a fraction whose denominator is a surd, to another that shall have a rational denominator without changing its value.

RULE. 1. *When the proposed fraction is a simple one, multiply each of its terms by the denominator.*

2. *If it be a compound surd, find such a multiplier by the last* Art. *as will make the denominator rational, then multiply both the numerator and denominator by it.*

EXAMPLES.

1. Reduce $\frac{b}{\sqrt{a}}$ to a fraction whose denominator shall be a rational. $\frac{b}{\sqrt{a}}\times\frac{\sqrt{a}}{\sqrt{a}}=\frac{b\sqrt{a}}{a}$. *Ans.*

2. Reduce $\frac{b}{\sqrt[3]{a}}$ to a fraction, whose denominator shall be rational. $\frac{b}{\sqrt[3]{a}}\times\frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}}=\frac{b\sqrt[3]{a^2}}{a}$. *Ans.*

3. Reduce the fraction $\frac{2}{\sqrt{5}}$ to another, whose denominator shall be rational.

$$\frac{2}{\sqrt{5}}=\frac{2}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}.\ \textit{Ans.}$$

4. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction, whose denominator shall be rational.

Here $\frac{3}{\sqrt{5}-\sqrt{2}}=\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=$

$$\frac{3\sqrt{5}+3\sqrt{2}}{3}=\frac{\sqrt{5}+\sqrt{2}}{1}=\sqrt{5}+\sqrt{2}.\ \textit{Ans.}$$

5. Extract the square root of $\frac{5}{8}$.

Here $\sqrt{\frac{5}{8}}=\frac{\sqrt{5}}{\sqrt{8}}=\frac{\sqrt{5}}{\sqrt{8}}\times\frac{\sqrt{8}}{\sqrt{8}}=\frac{\sqrt{40}}{\sqrt{64}}=\frac{2\sqrt{10}}{8}=\frac{\sqrt{10}}{4}$. *Ans.*

6. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction, whose denominator shall be rational.

Here $\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{3\sqrt{2}+2}{9-2}=\frac{3\sqrt{2}+2}{7}=$

$\frac{2}{7}+\frac{3}{7}\sqrt{2}$. *Ans*

7. Reduce $\frac{\sqrt{7}}{\sqrt{5}+\sqrt{3}}$ to a fraction, that shall have a rational denominator. *Ans.* $\frac{\sqrt{35}-\sqrt{21}}{2}$.

8. Reduce $\frac{\sqrt{3}}{3-\sqrt{1}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{\sqrt{3}}{2}$.

9. Reduce the fraction $\frac{5}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{5\sqrt{5}+5\sqrt{2}}{3}$.

10. Reduce the fraction $\frac{10}{\sqrt{5}+\sqrt{3}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{5\sqrt{5}-5\sqrt{3}}{1}$.

11. Reduce $\frac{1}{\sqrt{5}+\sqrt{7}}$ to a fraction that shall have a rational denominator. *Ans.* $\frac{\sqrt{5}-\sqrt{7}}{2}$.

12. Reduce the fraction $\frac{3}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction that shall have a rational numerator.

Ans. $\frac{3\sqrt{5}+3\sqrt{2}}{3}=\sqrt{5}+\sqrt{2}$.

SECTION XV.

QUADRATIC EQUATIONS, OR EQUATIONS OF THE SECOND DEGREE.

ART. 212. A quadratic equation is that in which the unknown quantity is of two dimensions, or which rises to the second power.

Quadratics are of two kinds. Those, which involve only the square of the unknown term, are called pure quadratics; and, those which involve both the first and second powers of the unknown quantity, are called affected quadratic equations.

The following are examples of pure quadratics.

EXAMPLES.

1. Given $4x^2-7=29$ to find x.

Conditions, $4x^2-7=29$;
Transposing, $4x^2=29+7=36$;
Dividing, $x^2=9$;
Extracting square root, $x=3$.

2. Given $ax^2+b=c$ to find x.

Conditions, $ax^2+b=c$
Transposing, $ax^2=c-b$
Dividing, $x^2=\frac{c-b}{a}$
Extracting square root, $x=\sqrt{\frac{c-b}{a}}$.

Hence, to find the value of the unknown term, we have the following

Rule. *Transpose and reduce the equation, so that the unknown quantity may be positive and the first term of the equation. Divide both terms of the equation by the coefficient of the unknown quantity; then extract the square root of both terms, and we obtain the value of the unknown quantity.*

3. Given $5x^2+5=3x^2+55$ to find x.

Conditions, $5x^2+5=3x^2+55$;
Transposing, $5x^2-3x^2=55-5$;
Reducing, $2x^2=50$;
Dividing, $x^2=25$;
Extracting square root, $x=5$.

4. Given $2x^2+8=3x^2-28$ to find x.

Conditions, $3x^2-28=2x^2+8$;
Transposing, $3x^2-2x^2=28+8$;
Reducing, $x^2=36$;
Extracting square root, $x=6$.

5. Given $7x^2-5=3x^2+11$ to find x. *Ans.* $x=\pm 2$.

6. Given $4x^2+15=7x^2-417$ to find x. *Ans.* $x=\pm 12$.

7. Given $3x^2+7=\frac{5x^2}{4}+35$ to find x. *Ans.* $x=\pm 4$.

8. Given $ax^2+n=m-c$ to find x.

Ans. $x=\sqrt{\frac{m-c-n}{a}}$.

9. Given $x^2-ab=d$ to find x. *Ans.* $x=\sqrt{d+ab}$.

10. A lady bought a silk dress for £8.15*s*, and the number of shillings she paid per yard was to the number of yards, as 4 to 7. How many yards did she purchase for her dress, and what was the price per yard?

Let $x=$ the number of shillings paid per yard.

Then $\frac{7x}{4}=$ the number of yards.

And the price of the whole, $\frac{7x^2}{4}=175$ shillings.
Clearing of fractions, $7x^2=700$;
Dividing, $x^2=100$;
Extracting the square root, $x=10s.$, price per yd.
Therefore, $\frac{7x}{4}=17\frac{1}{2}$ yards.

11. I have 10 acres of land. If it were a square field, what would be the length of one of its sides? *Ans.* 40 rods.

12. A and B lay out money on speculation. The amount of A's stock and gain is \$27, and he gains as much per cent. on his stock as B lays out. B's gain is \$32; and it appears, that A gains twice as much per cent. as B. Required the capital of each. *Ans.* A's capital \$15, B's \$80.

13. There are two square fields, the larger of which contains 25,600 square rods more than the other, and the proportion of their sides is as 5 to 3. Required the contents of each.

Ans. Contents of the larger, 40,000 square rods.
Contents of the smaller, 14,400 square rods.

14. I have three square house lots of equal size. If I were to add 193 square rods to their contents, it would make a square lot, whose sides would measure each 25 rods. Required the length of each of the sides of my three house lots.

Ans. 12 rods each.

15. A farmer has a square field, and the number of rods round it is $\frac{1}{10}$ the number of square rods of its contents. Required the number of acres in the field. *Ans.* 10 acres.

16. John Smith has a field, which is a right-angled parallelogram. Its sides are in the proportion of 4 to 3. A diagonal passing from one corner to its opposite is 100 rods. Required the contents of the field. *Ans.* 30 acres.

17. Two workmen, A and B, engage to work for a certain number of days at different rates. At the end of the time, A, who had been absent 4 days, received 75 shillings; but B, who had been absent 7 days, received only 48 shillings. Now, if B had been absent only 4 days and A 7 days, they would have received exactly alike. How many days were they engaged for? how many did each work, and what had each per day?

Ans. They were engaged to work 19 days. A worked 15 and B 12 days. A received 5 shillings, and B 4 shillings per day.

18. Two numbers are to each other as 4 to 5, and the sum of their cubes is 1512. What are those numbers?

Ans. 8 and 10.

19. A bushel measure contains $2150\frac{2}{5}$ cubic inches, and I wish to make a box that shall contain 50 bushels. Its length

is to be to its breadth as 3 to 1; and its height $\frac{3}{4}$ its breadth. What are its dimensions?

Ans. Length 108.84+, breadth 36.28+, and height 27.21+ inches.

20. What must be the dimensions of a cubical box that shall contain 100 bushels?

Ans. Height, length, and breadth, 59.9+ inches.

21. Two numbers are to each other as 3 to 7, and the difference of their cubes is 2528. What are those numbers?

Ans. 6 and 14.

22. Bought a house lot for $5184. Its length is to its breadth as 3 to 1. I gave as many dollars per square rod, as the lot is rods in breadth. What were the dimensions of the lot?

Ans. 36 rods long, 12 rods wide.

PROBLEMS.

23. Let m be divided into two quantities, whose squares shall be to each other as n to p.

Let $m =$ the greater,

And $m - x =$ the less.

Then $x^2 : (m-x)^2 :: n : p$.

Multiplying extremes, $px^2=n(m-x)^2$.

Evolution, $x\sqrt{p}=\pm\sqrt{n}(m-x)$.

Reducing, $x\sqrt{p}=m\sqrt{n}-x\sqrt{n}$.

Transposing, $x\sqrt{p}+x\sqrt{n}=m\sqrt{n}$.

Dividing, $x=\frac{m\sqrt{n}}{\sqrt{p}+\sqrt{n}}$, the greater.

Subtracting, $m-\frac{m\sqrt{n}}{\sqrt{p}+\sqrt{n}}=\frac{m\sqrt{p}}{\sqrt{p}+\sqrt{n}}$, the less.

If we take the minus sign we have

$$x\sqrt{p}=-\sqrt{n}(m-x).$$

Multiplying, $x\sqrt{p}=-m\sqrt{n}+\sqrt{n}.$

Transposing, $x\sqrt{p}-x\sqrt{n}=-m\sqrt{n}.$

Changing signs, $x\sqrt{n}-x\sqrt{p}=m\sqrt{n}.$

Dividing, $x=\dfrac{m\sqrt{n}}{\sqrt{n}-\sqrt{p}}$ the greater.

Subtracting, $m-\dfrac{m\sqrt{n}}{\sqrt{n}-\sqrt{p}}=\dfrac{-m\sqrt{p}}{\sqrt{n}-\sqrt{p}}$ the less.

24. Divide 18 into two such parts, that the square of the larger part shall be 25 times the square of the less.

Let $x=$ the larger; then $18-x=$ the less.

Then we have, $x^2 : (18-x)^2 : : 25 : 1.$

Multiplying extremes, $x^2=25(18-x)^2.$

Evolution, $x=5(18-x).$

Multiplying, $x=90-5x.$

Transposition, $6x=90.$

Dividing, $x=15$, the larger.

$18-15=3$, the less.

VERIFICATION.

$$15^2=25(3)^2.$$

Involving, $225=225.$

THE THEORY OF THE LIGHTS AND ATTRACTION.

213. To apply the foregoing problems, we premise the following principles of Natural Philosophy.

1. The intensity of light emanating from any luminous body, is inversely as the square of the distance from that body. That is, if the earth were twice the distance from the sun that it now is, it would receive only one-fourth part of the light

and heat that it now does. And, if it were removed at ten times the distance, it would have only one-hundredth part of the light and heat.

2. The quantity of light emanating from a celestial body is directly as the square of its diameter.

Hence, if the earth were four times the diameter of the moon, an inhabitant of that luminary would receive sixteen times as much light from the earth, as he would receive from the moon, if he were on the earth.

3. The laws of attraction are similar to those of light, for all bodies attract each other inversely as the squares of the distances from their centre, and directly as the mass of matter, which compose those bodies.

APPLICATION OF THE ABOVE PRINCIPLES.

25. The moon is 240,000 miles from the earth, and the quantity of matter in the earth is 80 times that of the moon. At what distance from the earth, in a direct line towards the moon, must a body be placed to be equally attracted by each, so that it will remain at rest, as it respects those bodies?

Let $d=$ the distance between the moon and earth.
$e=$ the quantity of matter in the earth.
$m=$ the quantity in the moon.
$x=$ the distance from the earth to the point required.
Then $d-x=$ the distance from the moon.

We have then the following proposition.

As $x^2 : (d-x)^2 : : e : m$.
Therefore, $mx^2=e(d-x)^2$.
By evolution, $x\sqrt{m}=\sqrt{e}(d-x)$.
Reducing, $x\sqrt{m}=d\sqrt{e}-x\sqrt{e}$.
Transposing, $x\sqrt{m}+x\sqrt{e}=d\sqrt{e}$.
Dividing, $x=\frac{d\sqrt{e}}{\sqrt{m}+\sqrt{e}}$.

Substituting the value of d, e, and m, we have

$x=\frac{240{,}000\sqrt{80}}{\sqrt{80}+\sqrt{1}}=\frac{2146624.8}{8{,}94427+1}=215865.4$ miles = the distance from the earth.

240,000—215865.4=24134.6 miles = the distance from the moon.

If we take the negative sign, we shall find the point *beyond* the moon, where the attraction of the two bodies will be equal.

Taking the minus sign, $x\sqrt{m}=-\sqrt{e}(d-x)$.

Reducing, $x\sqrt{m}=-d\sqrt{e}+x\sqrt{e}$.

Transposing, $x\sqrt{e}-x\sqrt{m}=d\sqrt{e}$.

Dividing, $x=\frac{d\sqrt{e}}{\sqrt{e}-\sqrt{m}}$.

Substituting the values of d, e, and m, we have

$x=\frac{2146624.8}{\sqrt{80}-\sqrt{1}}=270{,}210$ miles from the earth's centre, and therefore 270,210—240,000=30,210 miles beyond the moon.

26. Required the distance from the earth in a direction toward the sun, where a body would remain at rest, the distance of the earth being 95,000,000 miles from the sun, and the quantity of matter in the sun being 333,928 times greater than that of the earth.

Let S represent the quantity of matter in the sun.

E the quantity of matter in the earth, and D the distance between the earth and sun, and x the required distance from the sun.

Then substituting these letters for those in question 23, we have the following formula:

$$x=\frac{d\sqrt{s}}{\sqrt{s}+\sqrt{1}}=$$

$$\frac{95{,}000{,}000\sqrt{333{,}928}}{\sqrt{333{,}928}+\sqrt{1}}=94{,}835{,}885;$$

95,000,000—94,835,885=164,115 miles. *Ans.*

27. The diameter of Venus is 7700 miles, its distance from the sun is 68,000,000; the diameter of the earth is 7912 miles, and its distance from the sun as stated above is 95,000,000 miles. How much greater therefore is the intensity of light at Venusthan at the earth, and what is the comparative quantity that each receives from the sun? *Ans.*

28. Mercury is 37,000,000 miles from the sun. How much greater therefore is the intensity of light and heat at Mercury than at the earth. *Ans.* $6\frac{811}{1369}$ times.

29. Jupiter is 490,000,000 miles from the sun, and its diameter is 89,000 miles. Saturn is 900,000,000 miles from the sun, and its diameter is 79,000 miles. How much more light therefore do we receive from Jupiter than from Saturn when they are in opposition to the sun?

Let $a=$the distance of Jupiter from the sun.
$b=$the diameter of Jupiter.
$c=$his distance from the earth.
$d=$the distance of Saturn from the sun.
$e=$the diameter of Saturn.
$h=$his distance from the earth.

The distance of these planets from the earth is obtained by subtracting the earth's distance from the sun from their distance from the sun.

The surface of Jupiter is to the surface of Saturn as the squares of their diameter, and as the quantity of light which a planet receives from the sun is as the square of its diameter directly, and inversely as the squares of its distance from the sun.

Therefore if $b^2=$ the surface of Jupiter,
and $e^2=$ the surface of Saturn,
and a and d their respective distances from the sun.

Then the intensity of light at Saturn will be to the intensity of light at Jupiter as $\frac{e^2}{d^2}$ is to $\frac{b^2}{a^2}$. And as the light which each of these planets gives to the earth is in intensity inversely as the squares of their distances from the earth;

Therefore if $\frac{e^2}{d^2}=$ the quantity of light at Saturn, and $\frac{b^2}{a^2}=$ the quantity of light at Jupiter; then $\frac{e^2}{d^2h^2}=$ the quantity of light which Saturn gives to the earth, and $\frac{b^2}{a^2c^2}=$ the quantity which Jupiter gives.

Therefore, to find how much more light we receive from Jupiter than from Saturn, we use the following proportion.

$$\frac{e^2}{d^2h^2} : \frac{b^2}{a^2c^2} :: 1 : x.$$

Therefore, $$x=\frac{d^2h^2b^2}{e^2a^2c^2}.$$

If we substitute the value of these letters by numbers, we have

$$x=\frac{900^2\times805^2\times89^2}{79^2\times490^2\times395^2}=$$

$$\frac{810000\times648025\times7921}{6241\times240100\times156025}=17.7+.. \textit{ Ans.}$$

That is, we receive more than seventeen times as much light from Jupiter than we do from Saturn.

In the above operation, we have cancelled the cyphers in the distances and diameters of the planets.

AFFECTED QUADRATIC EQUATIONS.

214. An affected quadratic equation is one, which contains both the first and second power of the unknown quantity.

Every equation of this kind, having any real or positive root, will fall, when properly reduced, under one of the four following forms.

1. $x^2+ax=\ \ b$, where $x=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}$.

2. $x^2-ax=\ \ b$, where $x=+\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+b\right)}$.

3. $x^2+ax=-b$, where $x=-\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b\right)}$.

4. $x^2-ax=-b$, where $x=+\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}-b\right)}$.

215. No exact root can be taken of a binomial; but, if the first term of a binomial be a square of the unknown quantity, and the second term its root, with 1 or any other number or quantity for its coefficient, if half the square of the coefficient of the second term be added to the binomial, the quantity will then be an exact square. This may be illustrated by the following examples.

Let x^2+4x be the binomial, then 2 is half the coefficient of the second term, and its square is $2\times2=4$. This we add to the binomial, and the result is x^2+4x+4, and this quantity is an exact square, and its root by Art. 183 is $x+2$.

If the binomial be x^2+ax, and we add to it the square of half the coefficients $a=\frac{a^2}{4}$, the sum will be $x^2+ax+\frac{a^2}{4}$, the exact root of which is $x+\frac{a}{2}$:

Again, if the binomial be x^2-3abx, we have only to add the square of half the coefficient, $3ab=\left(\frac{3ab}{2}\right)^2=\frac{9a^2b^2}{4}$ to the binomial, and the sum will be an exact square, $x^2-3abx+\frac{9a^2b^2}{4}$.

For $\left(x^2-3abx+\frac{9a^2b^2}{4}\right)^{\frac{1}{2}}=x-\frac{3ab}{2}$.

216. If, therefore, there be any binomial, whose first term is an even power of the unknown quantity, and the second term half that power, and we add the square of half the coefficient of the second term to the binomial, the result will be an exact square.

217. To solve an affected quadratic equation, we adopt the following

RULE. *Bring all the unknown terms to one side of the equation, and the known terms to the other, observing so to arrange them, that the term, which contains the square of the unknown quantity, may be positive and stand first in the equation; and the term, which contains the first power of the unknown quantity, the second term of the equation.*

Divide each side of the equation by the coefficient of the unknown square, if it have a coefficient.

Add the square of half the coefficient of the second term to each side of the equation, and the unknown side will be a complete square.

Extract the square root of each side of the equation, and, from the result, the value of the unknown quantity may be obtained.

Given $x^2+8x=84$ to find the value of x.

Hereby the question, $x^2+8x=84$.

Completing the squares, $x^2+8x+16=84+16=100$.

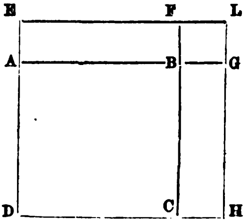
Extracting the square root, $x+4=10$.

Whence, $x=10-4$.

And, $x=6$. *Ans.*

In solving this question, we first add the square of half of 8, that is, 16 to both sides of the equation, we then extract the square root of $x^2+8x+16$, and find the result to be $x+4$; and the square root of $100=10$. Therefore, $x+4=10$; that is, $x=10-4=6$. *Ans.*

218. It may also be demonstrated by the following diagram, that, if the square of half the coefficient of the second term be added to the first member of an equation, it will be a complete square.



Let x represent one side of the square ABCD; then x^2 will represent this square. To this square we must add $8x$, and this quantity must be applied equally to the two sides AB and BC, or otherwise it would not be a square. Therefore $4x$, which is half of $8x$, will be applied to either side. If this quantity $4x$ be divided by x the quotient 4 will represent either of the distances EA or BG. Having added the two equal parallelograms EABF and BGHC to the square ABCD, we find our figure needs the small square FBGL to complete the square. The contents of this must be equal to the product of FB and BG, that is, 4 multiplied by 4, or the square of $4=16$; but 4 is half the coefficient of the second term. We add this quantity to x^2+8x, and the sum is $x^2+8x+16$, and its square root is $x+4$ by Art.

219. A quadratic may be solved by the following

RULE. *Having transposed the unknown terms to one side of the equation, and the known to the other, multiply each side by 4 times the coefficient of the unknown square.*

Add the square of the coefficient of the simple power of the unknown quantity to both sides of the equation, and the unknown side will then be a complete square.

Extract the root, and the value of the unknown quantity is obtained as before.

EXAMPLES.

1. Given $3x^2+4x-7=88$ to find the value of x.

Conditions, $3x^2+4x-7=88$.
Transposing, $3x^2+4x=88+7=95$.
Multiplying by 4 times 3, $36x^2+48x=1140$.
Completing the square, $36x^2+48x+16=1140+16=1156$.
Evolving, $6x+4=34$.
Transposing, $6x=30$.
Dividing, $x=5$. *Ans.*

2. Given $2x^2-10x+7=-5$ to find the value of x.

Conditions, $2x^2-10x+7=-5$.
Transposing, $2x^2-10x=-5+7=-12$.
Multiplying by 4 times 2, $16x^2-80x=-96$.
Completing the square, $16x^2-80x+100=-96+100=4$.
Evolving, $4x-10=2$.
Transposing, $4x=10+2=12$.
Dividing, $x=3$. *Ans.*

3. Given $3x^2+5x-8=34$ to find the values of x.
Ans. $x=3$, or $-4\frac{2}{3}$.

4. Given $x^2+6x+4=22-x$ to find the values of x.
Ans. $x=2$, or -9.

5. Given $8x^2-7x+6=171$ to find the values of x.
Ans. $x=5$, or $-\frac{33}{8}$.

6. Given $\frac{175x-350}{x}+10x-20=175$ to find the values of x.
Ans. $x=7$, or -5.

7. Given $x^2-6x+12=4$ to find the values of x.
Ans. $x=4$, or -2.

8. Given $8x^2+32x=360$ to find the value of x.

Conditions, $8x^2+32x=360$.
Dividing, $x^2+4x=45$.
Completing the square, $x^2+4x+4=45+4=49$.
Evolving, $x+2=7$.
Transposing, $x=7-2=5$, or $7+2=-9$.

9. Given $x^2-8x+50=98$ to find the values of x.

Conditions, $x^2-8x+50=98$.

Transposing, $x^2-8x=98-50=48$.

Completing the square, $x^2-8x+16=48+16=64$.

Evolving, $x-4=8$.

Transposing, $x=8+4=12$,

or $=8-4=-4$.

10. Given $x^2+ax=b$ to find the values of x.

Conditions, $x^2+ax=b$.

Completing the square, $x^2+ax+\frac{a^2}{4}=b+\frac{a^2}{4}$.

Evolving, $x+\frac{a}{2}=\sqrt{\left(b+\frac{a^2}{4}\right)}$.

Transposing, $x=\sqrt{\left(b+\frac{a^2}{4}\right)}\pm\frac{a}{2}$.

11. Given $3x^2-3x+6=5\frac{1}{3}$ to find the values of x.

Conditions, $3x^2-3x+6=5\frac{1}{3}$.

Transposing, $3x^2-3x=5\frac{1}{3}-6=-\frac{2}{3}$.

Reducing, $x^2-x=-\frac{2}{9}$.

Completing the square. $x^2-x+\frac{1}{4}=-\frac{2}{9}+\frac{1}{4}=-\frac{1}{36}$.

Evolving, $x-\frac{1}{2}=-\frac{1}{6}$.

Transposing, $x=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$,

or $=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$.

12. Given $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$ to find the values of x.

Conditions, $\frac{x^2}{2}-\frac{x}{3}+20\frac{1}{2}=42\frac{2}{3}$.

Transposing, $\frac{x^2}{2}-\frac{x}{3}=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$.

Clearing of fractions, $x-\frac{2x}{3}=44\frac{1}{3}$.

Completing the square, $x-\frac{2x}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=\frac{400}{9}.$

Evolving, $x-\frac{1}{3}=\frac{20}{3}=6\frac{2}{3}.$

Transposing, $x=6\frac{2}{3}+\frac{1}{3}=7,$

or $=6\frac{2}{3}-\frac{1}{3}=6\frac{1}{3}.$

13. Given $ax^2+bx=c$ to find the value of x.

Conditions, $ax^2+bx=c.$

Dividing, $x^2+\frac{bx}{a}=\frac{c}{a}.$

Completing the square, $x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}.$

Evolving and transposing, $x=\sqrt{\left(\frac{c}{a}+\frac{b^2}{4a^2}\right)}\mp\frac{b}{2a}.$

14. Given $ax^2-bx+c=d$ to find the value of x.

Conditions, $ax^2-bx+c=d.$

Transposing, $ax^2-bx=d-c$

Dividing, $x^2-\frac{bx}{a}=\frac{d-c}{a}.$

Completing the square, $x^2-\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{d-c}{a}+\frac{b^2}{4a^2}.$

Evolving, $x-\frac{b}{2a}=\sqrt{\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)}.$

Transposing, $x=\frac{b}{xa}\pm\sqrt{\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)}.$

Reducing, $x=\frac{b}{2a}\pm\frac{1}{2a}\sqrt{[4a(d-c)+b^2.]}$

220. If the equation contains two powers of the unknown quantity, and the exponent of the one is double that of the other, it may be resolved like a quadratic. Thus

15. Given $x^4+4x^2=117$ to find the value of x.

Conditions, $x^4+4x^2=117.$
Completing the square, $x^4+4x^2+4=117+4=121.$
Evolving, $x^2+2=11.$
Transposing, $x^2=11-2=9.$
Evolving, $x=3.$

16. Given $x^6-6x^3=16$ to find the value of x.

Conditions, $x^6-6x^3=16.$
Completing the square, $x^6-6x^3+9=16+9=25.$
Evolving, $x^3-3=5.$
Transposing, $x^3=5+3=8.$
Evolving, $x=2.$

17. Given $\frac{x}{2}-\frac{x^{\frac{1}{2}}}{3}=22\frac{1}{6}$ to find the value of x.

Conditions, $\frac{x}{2}-\frac{x^{\frac{1}{2}}}{3}=22\frac{1}{6}.$
Clearing of fractions, $x-\frac{2x^{\frac{1}{2}}}{3}=44\frac{1}{3}.$
Completing the square, $x-\frac{2x^{\frac{1}{2}}}{3}+\frac{1}{9}=44\frac{1}{3}+\frac{1}{9}=\frac{400}{9}.$
Evolving, $x^{\frac{1}{2}}-\frac{1}{3}=\frac{20}{3}.$
Transposing, $x^{\frac{1}{2}}=\frac{20}{3}+\frac{1}{3}=\frac{21}{3}=7.$
Involving, $x=49.$

18. Given $3x^{2n}-2x^n=25$ to find the value of x.

Conditions, $3x^{2n}-2x^n=25.$

Dividing, $x^{2n}-\frac{2x^n}{3}=\frac{25}{3}.$

Completing the square, $x^{2n}-\frac{2x^n}{3}+\frac{1}{9}=\frac{25}{3}+\frac{1}{9}=\frac{76}{9}.$

Evolving, $x^n-\frac{1}{3}=\frac{\sqrt{76}}{3}=\frac{2\sqrt{19}}{3}.$

Transposing, $x^n=\frac{1}{3}+\frac{2\sqrt{19}}{3}=\frac{1+2\sqrt{19}}{3}.$

Reducing, $x=\left(\frac{1+2\sqrt{19}}{3}\right)^{\frac{1}{n}}.$

19. Given $\sqrt{4x+16}=12$ to find the value of x.

Conditions, $\sqrt{4x+16}=12.$

Squaring both sides of the equation, $4x+16=144.$

Transposing, $4x=144-16=128.$

Dividing, $x=32.$

20. Given $\sqrt[3]{2x+3}+4=7$ to find the value of x.

Conditions, $\sqrt[3]{2x+3}+4=7.$

Transposing, $\sqrt[3]{2x+3}=7-4=3.$

Involving both sides, $2x+3=27.$

Transposing, $2x=27-3=24.$

Dividing, $x=12.$

21. Given $\sqrt{12+x}=2+\sqrt{x}$ to find the value of x.

Conditions, $\sqrt{12+x}=2+\sqrt{x}.$

Squaring both sides, $12+x=4+4\sqrt{x}+x.$

Transposing, &c. $8=4\sqrt{x}.$

Dividing, $2=\sqrt{x}.$

Involving, $4=x.$

22. Given $\sqrt{x+40}=10-\sqrt{x}$ to find the value of x.

Conditions, $\sqrt{x+40}=10-\sqrt{x}$.

Squaring both sides, $x+40=100-20\sqrt{x}+x$.

Transposing and reducing, $20\sqrt{x}=60$.

Dividing, $\sqrt{x}=3$.

Involving, $x=9$.

23. Given $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$ to find the value of x.

Conditions, $\sqrt{x-a}=\sqrt{x}-\frac{1}{2}\sqrt{a}$.

Involving, $x-a=x-\sqrt{ax}+\frac{a}{4}$.

Transposing, $\sqrt{ax}=a+\frac{a}{4}=\frac{5a}{4}$.

Involving, $ax=\frac{25a^2}{16}$.

Dividing by a, $x=\frac{25a}{16}$.

24. Given $3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$ to find the values of x.

Conditions, $3x^{\frac{4}{3}}-\frac{5x^{\frac{8}{3}}}{2}=-592$.

Changing the signs, &c. $\frac{5x^{\frac{8}{3}}}{2}-3x^{\frac{4}{3}}=592$.

Multiplying by $\frac{2}{5}$, $x^{\frac{8}{3}}-\frac{6x^{\frac{4}{3}}}{5}=\frac{1184}{5}$.

Completing the square, $x^{\frac{8}{3}}-\frac{6x^{\frac{4}{3}}}{5}+\frac{9}{24}=\frac{1184}{5}+\frac{9}{25}=\frac{5929}{25}$.

Extracting the root, $x^{\frac{4}{3}}-\frac{3}{5}=\pm\frac{77}{5}$

$x^{\frac{4}{3}}=\pm\frac{77}{5}+\frac{3}{5}=16$ or $-\frac{74}{5}$

$x=8$, or $-\left(\frac{74}{5}\right)^{\frac{3}{4}}$.

25. Given $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$ to find the values of x.

Conditions, $\sqrt{2x+1}+2\sqrt{x}=\frac{21}{\sqrt{2x+1}}$.

Clearing of fractions, $2x+1+2\sqrt{2x^2+x}=21$.

Transposition, $2\sqrt{2x^2+x}=20-2x$.

Division, $\sqrt{2x^2+x}=10-x$.

Squaring both sides, $2x^2+x=100-20x+x^2$.

Transposing, $x^2+21x=100$.

Completing the squares, $x^2+21x+\frac{441}{4}=100+\frac{441}{4}=\frac{841}{4}$.

Evolution, $x+\frac{21}{2}=\pm\frac{29}{2}$.

Transposition, $x=\pm\frac{29}{2}-\frac{21}{2}=4$, or -25.

26. Given $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$ to find the values of x.

Conditions, $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$.

Multiplying, $2x-2a+3\sqrt{2x^2-2ax}=7a+5x$.

Transposing, $3\sqrt{2x^2-2ax}=9a+3x$.

Dividing, $\sqrt{2x^2-2ax}=3a+x$.

Involving, $2x^2-2ax=9a^2+6ax+x^2$.

Transposing, $x^2-8ax=9a^2$.

Completing the squares, $x^2-8ax+16a^2=25a^2$.

Evolving, $x-4a=\pm 5a$.

Transposing, $x=\pm 5a+4a=9a$, or $-a$.

27. Given $x+5=\sqrt{x+5}+6$ to find the values of x.

Ans. $x=4$, or -1.

28. Given $\sqrt{5x+10}=\sqrt{5x}+2$ to find the value of x.

Conditions, $\sqrt{5x+10}=\sqrt{5x}+2.$

Squaring both sides, $5x+10=5x+4\sqrt{5x}+4.$

Transposing, &c. $6=4\sqrt{5x}.$

Dividing, $3=2\sqrt{5x}.$

Involving, $9=20x$

Dividing, &c. $x=\frac{9}{20}.$

29. Given $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}$ to find the value of x.

Conditions, $\frac{\sqrt{x}+2a}{\sqrt{x}+b}=\frac{\sqrt{x}+4a}{\sqrt{x}+3b}.$

Multiplying both sides of the equation by $\sqrt{x}+b$ and $\sqrt{x}+3b$, we have

$$x+(2a+3b)\times\sqrt{x}+6ab=x+(4a+b)\times\sqrt{x}+4ab.$$

Reducing, &c. $(2a-2b)\times\sqrt{x}=2ab.$

Dividing, $\sqrt{x}=\frac{ab}{a-b}$

Involving, $x=\left(\frac{ab}{a-b}\right)^2.$

30. Given $\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}$ to find the value of x.

$$\frac{1}{x}+\frac{1}{a}=\sqrt{\frac{1}{a^2}+\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}}.$$

Squaring both sides, $\frac{1}{x^2}+\frac{2}{ax}+\frac{1}{a^2}=\frac{1}{a^2}+\sqrt{\frac{4}{a^2a^2}+\frac{9}{x^4}}.$

Transposing, &c. $\frac{1}{x^2}+\frac{2}{ax}=\sqrt{\frac{4}{a^2x^2}+\frac{9}{x^4}}.$

Multiplying by x, $\frac{1}{x}+\frac{2}{a}=\sqrt{\frac{4}{a^2}+\frac{9}{x^2}}.$

Squaring both sides, $\frac{1}{x^2}+\frac{4}{ax}+\frac{4}{a^2}=\frac{4}{a^2}+\frac{9}{x^2}$.

Reducing, &c. $\frac{4}{ax}=\frac{8}{x^2}$.

Dividing, &c. $\frac{1}{a}=\frac{2}{x}$.

Transposing, &c. $x=2a$.

31. Given $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$ to find the value of x.

Ans. $x=\frac{b^2-4a^2}{4a}$.

32. Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$ to find the value of x.

Ans. $x=\frac{1}{1-a}$.

33. Given $x^2+12x-16=92$ to find the values of x.

Ans. $x=6$, or -18.

34. Given $x^2-3x=10$ to find the values of x.

Ans. $x=5$, or -2.

35. Given $x^2-x+3=45$ to find the values of x.

Ans. $x=7$, or -6.

36. Given $5x^2+x=4$ to find the values of x.

Ans. $x=\frac{4}{5}$, or -1.

37. Given $2x^2-x=21$ to find the values of x.

Ans. $x=\frac{7}{2}$, or -3.

38. Given $5x^2+6x-3=60$ to find the values of x.

Ans. $x=3$, or $-\frac{21}{5}$.

39. Given $(x-12)(x+2)=0$ to find the values of x.

Ans. $x=12$, or -2.

40. Given $3x^2-14x+15=0$ to find the values of x.

Ans. $x=3$, or $1\frac{2}{3}$.

41. Given $ax^2-bx=c$ to find the value of x.

$$\text{. } \textit{Ans.}\quad x=\frac{b\pm\sqrt{(b^2+4ac)}}{2a}.$$

42. Given $4x^2-6x=108$ to find the values of x.

Ans. $x=6$ or $-4\frac{1}{2}$.

43. Given $4x-\frac{14-x}{x+1}=14$ to find the values of x.

Ans. $x=4$, or $-\frac{7}{4}$.

44. Given $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$ to find the values of x.

Ans. $x=3$, or $\frac{21}{11}$.

45. Given $x+\sqrt{5x+10}=8$ to find the values of x.

Ans. $x=18$, or 3.

46. Given $x+\sqrt{10x+6}=9$ to find the values of x.

Ans. $x=25$, or 3.

47. Given $3x^2+2x-9=76$ to find the value of x.

Ans. $x=5$.

48. Given $x^2-10x=-25$ to find the value of x.

Ans. $x=5$.

49. Given $3x^2-x-140=0$ to find the value of x.

Ans. $x=7$.

50. Given $5x^2+\frac{7x}{2}=7x^2-51$ to find the value of x.

Ans. $x=6$.

51. Given $2x^2-\frac{4x-4}{3}=7x$ to find the value of x.

Ans. $x=4$.

52. Given $\frac{x^2}{5}+20x=3x^2-80$ to find the value of x.

Ans. $x=10$.

53. If $x^2+8x=65$, what are the two values of x?
Ans. $x=5$, or -13.

54. If $6x^2-x=92$, what are the two values of x?
Ans. $x=4$, or $-\frac{23}{6}$.

55. If $3x^2+4x=340$, what are the two values of x?
Ans. $x=10$, or $-11\frac{1}{3}$.

56. If $x^2-10x=-21$, what are the two values of x?
Ans. $x=7$, or 3.

57. If $5x^2-\frac{x}{2}=78$, what are the two values of x?
Ans. $x=4$, or $-3\frac{9}{10}$.

58. If $11x^2-100x=-201$, what are the two values of x?
Ans. $x=3$, or $6\frac{1}{11}$.

59. If $3x^2-17x=2x^2+84$, what are the two values of x?
Ans. $x=21$, or -4.

60. Given $x+16-7\sqrt{x+16}=10-4\sqrt{x+16}$ to find the values of x.
Ans. $x=9$, or -12.

61. Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find the values of x.
Ans. $x=\frac{4}{3}$, or $-\frac{1}{3}$.

62. Given $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$ to find the values of x.
Ans. $x=9$, or 4.

63. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$ to find the value of x.
Ans. $x=\pm a\sqrt{\frac{1\pm\sqrt{5}}{2}}$.

64. Given $x-1=2+\frac{2}{x^{\frac{1}{2}}}$ to find the values of x.
Ans. $x=4$, or -1.

65. Given $\sqrt[3]{x^3-a^3}=x-b$ to find the value of x.

Ans. $x=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}$.

66. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$ to find the values of x.

Ans. $x=4$, or $-\frac{64}{9}$.

67. Given $\sqrt{x^3}-2\sqrt{x}-x=0\sqrt{x}$ to find the values of x.

Ans. $x=4$, or 1.

68. Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$ to find the values of x.

Ans. $x=2$, or -3.

69. Given $\frac{x}{2}=22\frac{1}{6}+\frac{\sqrt{x}}{3}$ to find the values of x.

Ans. $x=49$, or $\frac{361}{9}$.

70. Given $\frac{\frac{3\sqrt{x}-2}{5}}{x-5}-\frac{1}{20}=0$ to find the values of x.

Ans. $x=49$, or 25.

71. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=746$ to find the values of x.

Ans. $x=253$, or $-28^{\frac{5}{3}}$.

72. Given $x^3-x^{\frac{3}{2}}=56$ to find the values of x.

Ans. $x=4$, or $\sqrt[3]{49}$.

73. Given $\sqrt{2x-1}+2\sqrt{x}=\frac{21}{\sqrt{2x-1}}$ to find the values of x.

Ans. $x=4$, or -25.

74. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$ to find the values of x.

Ans. 4, or 69.

PROBLEMS.

1. A merchant bought a number of pieces of two kinds of silk for £92 3*s*. There were as many pieces bought of each kind, and as many shillings paid per yard for them as a piece of that kind contained yards. Now two pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Let $x=$ the number of yards in one piece, and therefore it will equal the number of pieces, and also the number of shillings per yard; and $19-x=$ the number of yards in the other piece.

Therefore $x^3+(19-x)^3=$ the value of both kinds,

And $x^3+(19-x)^3=1843$,

Or $57x^2-1083x+6859=1843$.

By transposition, $57x^2-1083x=-5016$,

Or $x^2-19x=-88$.

Completing the square, $x^2-19x+\frac{361}{4}=\frac{361}{4}-88=\frac{9}{4}$.

Evolution, $x-\frac{19}{2}=\pm\frac{3}{2}$,

$$x=\pm\frac{3}{2}+\frac{19}{2}=11 \text{ or } 8.$$

$$19-=8 \text{ or } 11.$$

Both values answer the conditions of the question, therefore there were 11 yards in one, and 8 in the other.

2. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame all parts of which are of equal width, and whose area is to be equal to that of the glass. Required the width of the frame. *Ans.* 3 inches.

3. A grazier bought as many sheep as cost him £60; out of which he reserved 15, and sold the remainder for £54, gaining two shillings a head by them. How many sheep did he buy, and what was the price of each?

Ans. 75 sheep at 16 shillings each.

4. A merchant sold a quantity of flour for $39, and gained as much per cent. as the flour cost him. What was the price of the flour? *Ans.* $30.

5. There are two numbers, whose difference is 9, and their sum multiplied by the greater is 266. What are those numbers? *Ans.* 14 and 5.

6. A and B gained by trade $18. A's money was in the firm 12 months, and he received for his principal and gain $26. B's money, which was $30, was in the firm 16 months. What money did A put into the firm? *Ans.* 20.

7. A merchant bought a quantity of flour for $72, and he found that if he had bought 6 barrels more for the same money, he would have paid $1 less for each barrel. How many barrels did he buy, and what was the price of each? *Ans.* He bought 18 barrels, at $4 per barrel.

8. A square court-yard has a gravel walk around it. The side of the court wants 2 yards of being 6 times the breadth of the gravel walk; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 92 yards. Required the area of the court. *Ans.* 256 square yards.

9. Given $\frac{1+x^3}{(1+x)^3}=\frac{1}{3}$ to find the values of x. *Ans.* $x=2$ or $\frac{1}{2}$.

10. Given $x^4-2x^3+x=132$ to find the value of x. *Ans.* $x=1\pm\frac{\sqrt{-43}}{2}$.

11. Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$ to find the values of x. *Ans.* $x=\frac{4}{3}$ or $-\frac{1}{3}$.

12. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225. *Ans.* 9 and 12.

221. Many questions in cubic equations may be performed by quadratic equations.

EXAMPLES.

1. What is the value of x in the following equation $x^3-x^2=4$?

$$x^3-x^2=4.$$

Multiplying by 4, $4x^3-4x^2=16.$

Adding x^4 to each side, $x^4+4x^3-4x^2=x^4+16.$

Transposing, $x^4+4x^3=x^4+4x^2+16.$

Adding $4x^2$ to each side, $x^4+4x^3+4x^2=x^4+8x^2+16.$

Evolving, $x^2+2x=x^2+4.$

Reducing, $2x=4.$

Dividing by 2, $x=2.$

2. What is the value of x in the following equation $x^3-3x=2$?

$$x^3-3x=2.$$

Multiplying by x, $x^4-3x^2=2x.$

Adding x^2, $x^4-2x^2=x^2+2x.$

Completing the squares, $x^4-2x^2+1=x^2+2x+1.$

Evolving, $x^2-1=x+1.$

Transposing, $x^2-x=2.$

Completing the squares, $x^2-x+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}.$

Evolving, $x-\frac{1}{2}=\frac{3}{2}.$

Transposing, $x=\frac{3}{2}\pm\frac{1}{2}=2$ or $-1.$

3. There is a cubical block of marble, and if 50 be added to the number of square feet in its surface, it would be equal to the number of cubic feet in its contents. What are the solid contents of the block?

Let $x=$ the side of the cube.

Then $x^3=$ the contents of the block.

And $x^2=$ the superficial contents of one side of the block.

Then $3x^2=$ the superficial contents of one-half the surface of the block.

Therefore $x^3=3x^2+50.$

Adding $2x^2$ to both sides, $x^3+2x^2=5x^2+50.$

Multiplying both sides by 8, $8x^3+16x^2=40x^2+400.$

Adding x^4, $x^4+8x^3+16x^2=x^4+40x^2+400.$

Evolving, $x^2+4x=x^2+20.$

Reducing, $4x=20.$

Dividing, $x=5.$

Therefore the contents, $x^3=125$ cubic feet.

4. A gentleman having asked a lady her age, she replied, that if 29 times the square of her age were subtracted from twice its cube, the remainder would be 225. What was the lady's age?

Let $x=$ her age.

Then, $2x^3-29x^2=225.$

Adding x^4 and x^2, $x^4+2x^3-29x^2+x^2=x^4+x^2+225.$

$$x^4+2x^3+x^2=x^4+30x^2+225.$$

$$x^2+x \quad =x^2+15.$$

$$x \quad =15 \text{ years.}$$

5. A boy being asked what he gave for his books replied, that if 51 times the square ~~sum he gave square~~ of the number of dollars he gave for them were subtracted from 6 times the cube of the number, the remainder would be 900. What was the price of the books?

QUADRATICS WITH TWO OR MORE UNKNOWN TERMS.

1. Given $\left.\begin{array}{l} x+y=10 \\ xy=16 \end{array}\right\}$ (And) to find the values of x and y.

(1.) First equation, $x+y=10.$

(2.) Second equation, $xy=16.$

(3.) Squaring the 1st., $x^2+2xy+y^2=100.$

(4.) Multiplying (2) by 4, $4xy \quad =64.$

(5.) Subtracting 4th from 3d, $x^2-2xy+y^2=36.$

(6.) Evolving 5th, $x-y=\pm 6.$

(7.) The 1st, $x+y=10$.
(8.) Adding 6th and 7th, $2x=16$ or 4.
(9.) Subtracting 6th from 7th, $2y=4$ or 16.
(10.) Dividing the 8th by 2, $x=8$ or 2.
(11.) Dividing the 9th by 2, $y=2$ or 8.
Hence $x=8$ or 2, and $y=2$ or 8.

This method may be adopted whenever the sum and product of two unknown quantities are given.

2. $\left.\begin{array}{ll}\text{Given} & x-y=3 \\ \text{And} & xy=10\end{array}\right\}$ to find the values of x and y.

(1.) First condition, $x-y=3$.
(2.) Second condition, $xy=10$.
(3.) Squaring 1st, $x^2-2xy+y^2=9$.
(4.) Multiplying 2d by 4, $4xy=40$.
(5.) Adding 3d and 4th, $x^2+2xy+y^2=49$.
(6.) Evolving the 5th. $x+y=\pm 7$.
(7.) The 1st, $x-y=3$.
(8.) Adding 6th and 7th, $2x=10$ or -4.
(9.) Dividing 8th by 2, $x=5$ or -2.
(10.) Subtracting 7th from 6th, $2y=4$ or -10.
(11.) Dividing 10th by 2, $y=2$ or -5.
Hence $x=5$ or -2 and $y=2$ or -5.

We may proceed in the same manner, whenever the difference and product of two unknown quantities are given.

3. $\left.\begin{array}{ll}\text{Given} & x+y=20 \\ \text{And} & x^2+y^2=208\end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x+y=20$.
(2.) Second equation, $x^2+y^2=208$.
(3.) 2d multiplied by 2, $2x^2+2y^2=416$.

(4.) Square of the 1st, $x^2+2xy+y^2=400.$

(5.) Subtracting 4th from 3d, $x^2-2xy+y^2=16.$

(6.) Evolving 5th, $x-y=\pm 4.$

(7.) First equation, $x+y=20.$

(8.) Sum of 6th and 7th, $2x=24$ or $16.$

(9.) Half of the 8th, $x=12$ or $8.$

(10.) Subtracting 6th from 7th, $2y=16$ or $24.$

(11.) Half of 10th, $y=8$ or $12.$

Hence $x=12$ or 8; $y=8$ or $12.$

4. $\left.\begin{array}{ll}\text{Given} & x-y=3 \\ \text{And} & x^2+y^2=117\end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x-y=3.$

(2.) Second equation, $x^2+y^2=117.$

(3.) The 2d multiplied by 2, $2x^2+2y^2=234.$

(4.) Square of the 1st, $x^2-2xy+y^2=9.$

(5.) Subtracting 4th from 3d, $x^2+2xy+y^2=225.$

(6.) Evolving the 5th, $x+y=\pm 15.$

(7.) The 1st, $x-y=3.$

(8.) Sum of the 6th and 7th, $2x=18$ or $-12.$

(9.) Dividing 8th by 2, $x=9$ or $-6.$

(10.) Subtracting 7th from 6th, $2y=12$ or $-18.$

(11.) Dividing 10th by 2, $y=6$ or $-9.$

Hence $x=9$ or -6; $y=6$ or $-9.$

5. $\left.\begin{array}{ll}\text{Given} & \sqrt{x^2+y^2}=10 \\ \text{And} & x^2-y^2=28\end{array}\right\}$ to find the values of x and y.

(1.) First equation, $\sqrt{x^2+y^2}=10.$

(2.) Second equation, $x^2-y^2=28.$

(3.) Square of the 1st, $x^2+y^2=100.$

(4.) Sum of 2d and 3d, $2x^2=128.$

(5.) Half the 4th, $x^2=64$.
(6.) Square root of 5th, $x=8$.
(7.) Subtract 2d from 3d, $2y^2=72$.
(8.) Half the 7th, $y^2=36$.
(9.) Square root of 8th, $y=6$.
Hence $x=8$, and $y=6$.

6. $\left.\begin{array}{ll}\text{Given} & x+y=5 \\ \text{And} & x^3+y^3=25\end{array}\right\}$ to find the values of x and y.

(1.) First equation, $x+y=5$.
(2.) Second equation, $x^3+y^3=35$.
(3.) Square of the 1st, $x^2+2xy+y^2=25$.
(4.) The 2d divided by the 1st, $x^2-xy+y^2=7$.
(5.) Subtracting 4th from 3d, $3xy=18$.
(6.) Dividing 5th by 3, $xy=6$.
(7.) The 4th, $x^2-xy+y^2=7$.
(8.) The 6th, $xy=6$.
(9.) Subtracting 6th from 7th, $x^2-2xy+y^2=1$.
(10.) Evolving the 9th, $x-y=1$.
(11.) The 1st, $x+y=5$.
(12.) Sum of 10th and 11th, $2x=6$.
(13.) Half of 12th, $x=3$.
(14.) Subtracting 10th from 11th, $2y=4$.
(15.) Half of 14th, $y=2$.
Hence $x=3$, and $y=2$.

7. $\left.\begin{array}{ll}\text{Given} & x^2+y^2=20 \\ \text{And} & x^2-y^2=12\end{array}\right\}$ to find the values of x and y.

Ans. $x=4$; $y=2$.

8. $\left.\begin{array}{ll}\text{Given} & x+y=6 \\ \text{And} & x^2+y^2=26\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$, and $y=1$.

9. $\left.\begin{array}{l}\text{Given } x^2+y^2=74 \\ \text{And } x-y=2\end{array}\right\}$ to find the values of x and y.

Ans. $x=7$, and $y=5$.

10. $\left.\begin{array}{l}\text{Given } x^2+y^2=149 \\ \text{And } x+y=17\end{array}\right\}$ to find the values of x and y.

Ans. $x=10$, and $y=7$.

11. $\left.\begin{array}{l}\text{Given } x^2-y^2=85 \\ \text{And } x+y=17\end{array}\right\}$ to find the values of x and y.

Ans. $x=11$ and $y=6$.

12. $\left.\begin{array}{l}\text{Given } x-y=2 \\ \text{And } x^3-y^3=98\end{array}\right\}$ to find the values of x and y.

Ans. $x=5$ and $y=3$.

13. $\left.\begin{array}{l}\text{Given } 10x+y=3xy \\ \text{And } y-x=2\end{array}\right\}$ to find the values of x and y.

Ans. $x=2$ or $-\frac{1}{3}$ and $y=4$ or $-\frac{5}{3}$.

EXAMPLES.

1. A says to B, the sum of our money is 18 dollars. B replies, but if twice the number of your dollars were multiplied by mine, the product would be $154. How many dollars had each? *Ans.* A had $7, and B had $11.

2. The difference of two numbers is 5, and the sum of their squares is 193. What are those numbers? *Ans.* 12 and 7.

3. A and B have each a small field; and each of which is an exact square; and it requires 200 rods of fence to inclose both. The contents of these fields are 1300 square rods. What is the value of each at $2.25 per square rod?

Ans. A's field $900. B's $2025.

4. A lady wishes to purchase a carpet for each of her square parlors; one of which is 3 feet longer than the other, and it will require 85 square yards for both rooms. Mr. Ames has good carpeting, which is 40 inches wide, which he will sell at $1.75 per yard. What will it cost the lady to carpet each of her rooms?

Ans. For the larger room, $77.17½; smaller, $56.70.

5. There are two piles of wood, each of which is a perfect cube. The sum of their lengths is 20 feet, and their contents are 2240 cubic feet. What is the value of each pile at $6.25 per cord?

Ans. Value of the larger pile, $84.37½; the smaller, $25.

6. There are two square buildings that are paved with stones a foot square each. The circumference of the larger building exceeds that of the smaller by 48 feet, and both their pavements together contain 2120 stones. What are the lengths separately? *Ans.* 26 and 38 respectively.

7. A sets out from Boston for Portland, the distance being 105 miles. B sets out at the same time from Portland for Boston. A arrives in Portland in 9 hours, B arrives in Boston in 16 hours after they meet. In what time does each perform the journey? *Ans.* A in 21 hours. B in 28 hours.

8. Divide 60 into two such parts that their product shall be to the difference of their squares as 2 to 3. *Ans.* 40 and 20.

9. There are two numbers whose product is 77, and the difference of whose squares is to the squares of their difference as 2 to 9. Required the numbers. *Ans.* 11 and 7.

10. I have two house lots, the contents of which are 225 square rods, and the square rods in the less are to the square rods in the larger as 4 to 27. Required the contents of each lot. *Ans.* 81 square rods in the less, and 144 in the larger.

11. The product of two numbers is 48, and the difference of their cubes is to the cube of their difference as 37 to 1. Required the numbers. *Ans.* 8 and 6.

12. There are two numbers whose product is 196; and if the greater be divided by the less, the quotient is 4. What are those numbers? *Ans.* 28 and 7.

13. A, B and C can perform a piece of work in a certain time; A can perform it in 6 hours, B in 15 hours, and C in 10 hours; how long would it take them all to perform it?
Ans. 3 hours.

14. A grazier bought a certain number of oxen for $240, and having lost 3, he sold the remainder at $8 a head more than they cost him, thus gaining $59 by his bargain; what number did he buy? *Ans.* 16.

15. The paving of two court-yards cost £205; a square yard of each cost $\frac{1}{4}$ as many shillings as there were yards in a side of the other. And a side of the greater and less together measure 41 yards. Required the length of a side of each?
Ans. 25 and 16 yards.

16. Divide 145 into two such parts that the sum of their square roots shall be 17. *Ans.* 81 and 64.

17. Sold an ox for $56 and gained as much per cent. as the ox cost. What was paid for him? *Ans.* $40.

18. Divide the number 14 into two factors so that the sum of their cubes shall be 728. *Ans.* 8 and 6.

19. My farm is a parallelogram, and the length is twice its breadth; but, having enlarged it two rods on each side, I find its contents increased 496 square rods. Of how many acres does my farm at present consist? *Ans.* 23 acres, 16 rods.

20. There are two numbers whose product added to the sum of their squares is 109; but the difference of whose squares is 24. Required those numbers. *Ans.* 5 and 7.

21. What number is that, to which if 40 be added, and the square root extracted, this root shall be less than the original quantity by 16? *Ans.* 24.

22. Two gentlemen, A and B, speaking of their ages, A said that the product of their ages was 750. B replied that if his age was increased by 7 years and A's was lessened by 2 years their product would be 851. Required their ages.

Ans. A's 25, and B's 30 years.

23. John Smith's garden is an oblong, and contains 15,000 square yards; and he, being a man of taste, surrounded it by a walk 7 yards wide, the contents of which were 3696 square yards. Required the length and breadth of the garden.

Ans. Length 150, breadth 100 yards.

24. A gentleman purchased a farm for $5600; but if his farm had contained 10 acres more, it would have cost him $10 less per acre. Of how many acres did his farm consist?

Ans. 70 acres.

25. A man purchased a farm in the form of a parallelogram whose length was four times its breadth. It cost $\frac{1}{4}$ as many dollars per acre as the field was rods in length, and the number of dollars paid for the farm was four times the number of rods round it. Required the price of the farm, and its length and breadth.

Ans. Price $1600. Length 160 rods, breadth 40 rods.

26. Two men, A and B, set out from the same place at the same time to travel to Boston, it being 39 miles distant. A travelled $\frac{1}{4}$ of a mile an hour faster than B, and arrived at Boston an hour sooner. Required the rates of travelling.

Ans. A $3\frac{1}{4}$, and B 3 miles per hour.

27. It is required to find four numbers in geometrical progression, such that the difference of the two means shall be 14, and the difference of the extremes 49.

Ans. 7, 14, 28 and 56.

28. A certain company agreed to build a vessel for $6300; but two of their number having died, those that survived had

each to advance $200 more than they otherwise would have done. Of how many persons did the company at first consist?

Ans. 9 persons.

29. I have an oblong field of corn which consists of 6250 hills, but the number of hills in the length exceed those of the breadth by 75. Of how many hills did the length and breadth consist? *Ans.* 125 hills the length, 50 the breadth.

30. A man bought 10 ducks and 12 turkeys for $22.50. He had 4 more ducks for $6 than he had of turkeys for $5. What was the price of each?

Ans. The price of a duck was 75 cents, and of a turkey $1.25.

31. What number is that, to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18? *Ans.* 25.

32. A has two gardens, each of which is an exact square. They contain 208 square rods. It requires 80 rods of fence to enclose both gardens. Required the contents of each.

Ans. 144 square rods. 64 square rods.

33. A has two square gardens, and it requires 80 rods of fence to enclose them. The larger contains 80 square rods more than the other. How many square rods do both gardens contain? *Ans.* 208 square rods.

34. A has two square gardens, and the side of the one exceeds that of the other by 4 rods, and the contents of both are 208 square rods. How many square rods does the larger garden contain more than the smaller?

Ans. 80 square rods.

35. I have two blocks of marble which are exact cubes, and whose united lengths are 20 inches, and they contain 2240 cubic inches. Required the surface of each.

Ans. Larger, 864 inches. Smaller, 384 inches.

36. A merchant sold a bale of cloth for $50, and gained as much per cent. as the cloth cost him. What was the price of the cloth? *Ans.* $75.

37. There are two numbers whose difference is 12; and their sum multiplied by the greater is 560. What are those numbers? *Ans.* 20 and 8.

38. The plate of a looking-glass is 36 inches by 12 inches. It is to be framed with a frame of equal width, whose area is 448 square inches. What is the width of the frame? *Ans.* 4 inches.

39. Divide 100 into two such parts that the sum of their square roots may be 14. *Ans.* 64 and 36.

40. A square court-yard has a rectangular gravel walk around it. The side of the court wants one yard of being six times the breadth of the gravel walk, and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 340. What is the area of the court and width of the walk?

Ans. Area of the court 529 square yards. Width of the walk 4 yards.

41. A merchant bought 54 bushels of wheat, and a certain quantity of barley. For the former he gave half as many shillings per bushel as there were bushels of barley, and for the latter 4 shillings per bushel less. He sold the mixture at 10 shillings per bushel and lost £28 16*s*. by his bargain. What was the price of the barley?

Ans. 36 bushels of barley at 14 shillings per bushel.

42. I have 165½ square feet of plank 3 inches in thickness, with which I intend to make a cubical box. Required its contents in cubic feet. *Ans.* 125 cubic feet.

43. I have a small globule of glass one inch in diameter. How large a sphere may be made of it, if the glass is to be only $\frac{1}{20}$ of an inch in thickness: and taking it for *granted*, that all spheres are to each other as the cubes of their diameters.

Ans.

44. A and B can dig a cellar in a days. A and C can do the labor in b days, and B and C can do the same in c days. In what time would each perform the labor, and how long would it require A, B, and C to complete the work? *Ans.*

45. Two men, A and B, bought a farm of 200 acres, for which they paid $200 each. On dividing the land, A says to B, if you will let me have my part in the situation, which I shall choose, you shall have so much more land than I, that mine shall cost 75 cents per acre more than yours. B accepted the proposal. How much land did each have, and what was the price of each per acre? *Ans.*

SECTION XVI.

CUBIC AND HIGHER EQUATIONS.

ART. **222.** A Cubic Equation, or equation of the third power, is one that contains the third power of the unknown quantity,

As $x^3-ax^2+bx=c$.

223. A Biquadratic is an equation, that contains the fourth power of the unknown quantity,

As $x^4-ax^3+bx^2-cx=d$.

224. An equation of the fifth power is one, that contains the fifth power of the unknown quantity,

As $x^5-ax^4+bx^3-cx^2+dx=e$.

And so on for all other higher powers.

There are many particular and very prolix rules given for the solution of the above-mentioned powers or equations. But they all may be readily solved by the following easy

RULE. 1. *Find by trial, two numbers as near the true root as convenient, and substitute them separately in the given equation, instead of the unknown quantity; and find how much the terms collected together, according to their signs* + or —, *differ from the absolute known term of the equation, noting whether these errors are in excess or defect.*

2. *Multiply the difference of the two numbers, found or taken by trial, by either of the errors, and divide the product by the difference of the errors, when they are alike, but by their sum, when they are unlike. Or we may say, as the difference or sum of the errors, is to the difference of the two numbers, so is either error to the correction of its supposed number.*

3. *Add the quotient last found to the number belonging to that error, when its supposed number is too little, but subtract it when too great, and the result will give the true root* nearly.

4. *Take this root and the nearest of the two former, or any other that may be found nearer; and, by proceeding in like manner as above, a root will be obtained nearer than before. Proceeding in the same manner, we may obtain the answer to any degree of exactness required.*

NOTE 1.—It is best always to employ two assumed numbers, that shall differ from each other only by unity in the last figure on the right; because then the difference, or multiplier, is only 1. It is also best to use always the least error in the above operation.

NOTE 2.—It will be convenient also to begin with a single figure at first, trying several single figures, till there be found the two nearest the truth, the one too little, and the other too great; and in working with them, find only one more figure. Then substitute this corrected result in the equation for the unknown letter; and, if the result prove too little, substitute also the number next greater for the second supposition; but, if the former prove too great, then take the next less number for the second supposition, and working with the second pair of errors, continue the quotient only so far as to have the corrected number to four places of figures. Then repeat the same process again with this last corrected number, and the next greater or less, as the case may require, carrying the third corrected number to eight figures, because each new operation commonly doubles the number of true figures. Proceed in this manner to any extent that may be wanted.

EXAMPLES.

1. Find the root of the cubic equation $x^3+x^2+x=100$.

We see that x lies between 4 and 5. We assume, therefore, 4 and 5 as the two values of x.

FIRST SUPPOSITION.				SECOND SUPPOSITION.
4	=	x	=	5
16	=	x^2	=	25
64	=	x^3	=	125
84		sums		155
100		but should be		100
—16		errors		+55

Sum of the errors, $55+16=71$.

Then, 71 : 1 : : 16 : .2.

Hence, $x=4+.2=4.2$ nearly.

Again, let $x=4.2$ and 4.3.

FIRST SUPPOSITION.		SECOND SUPPOSITION.
4.2	x	4.3
17.64	x^2	18.49
74.088	x^3	79.507
95.928	sums	102.297
100		100
—4.072		+2.297

Sum of the errors, $4.072+2.297=6.369$.

As 6.369 : .1 : : 2.297 : 0.036.

Hence $x=4.3—.036=4.264$ nearly.

Again, let $x=4.264$ and 4.265.

FIRST SUPPOSITION.		SECOND SUPPOSITION.
4.264	x	4.265
18.181696	x^2	18.190225
77.526752	x^3	77.581310
99.972448		100.036535
100		100
0.027552		0.036535

Sum of the errors, $.027552+.036535=.064087$.

As $.064087 : .001 :: .027552 : 0.0004299$.

Hence, $x=4.264+.0004299=4.2644299$ nearly.

2. Find the root of the equation $x^3-15x^2+63x=50$.

Here it is evident, that the root is more than 1. We then assume the two values of x to be 1.0 and 1.1.

Then	63.0	=	$63x$	=	69.3
	-15	=	$-15x^2$		-18.15
	1	=	x^3		1.331
	49		sums		52.481
	50				50
	-1		errors		$+2.481$

Sum of the errors, $1+2.481=3.481$.

As $3.481 : 1 :: 1 : .03$

Add 1.00

Hence x = 1.03 nearly.

Again, let $x=1.03$ and 1.02.

Then	64.89	$63x$	64.26
	-15.9135	$-15x^2$	15.6060
	1.092727	x^3	1.061208
	50.069227	sums	49.715208
	50		50
	$+.069227$	errors	$-.284792$
	$.284792$		

$$.354019 : .01 :: .069227 : .0019555.$$

Hence $x=1.03-.0019555=1.02804$ nearly.

3. Find the value of x in the equation $x^3+10x^2+5x=260$.
Ans. $x=4.1179857$.

4. Find the value of x in the equation $x^3-2x=50$.
Ans. $x=3.8648854$.

5. Find the value of x in the equation $x^4-3x^2-75x=10000$.
Ans. $x=10.2609$.

6. Find the value of x in the equation $x^5+2x^4+3x^3+4x^2+5x=54321$.
Ans. $x=8.414455$.

SECTION XVII.

RATIOS.

Art. 225. Ratio is the relation which one quantity bears to another of a similar kind with respect to its magnitude.

226. The magnitude or value of a ratio is estimated by stating how often one quantity or number contains or is contained in another. Thus, in comparing 16 with 2, we observe that it has a certain relative magnitude with respect to 2, which it contains 8 times; and, if we compare 16 with 4, we observe that it has a different relative magnitude, for it contains 4 only 4 times. Hence, 16 is less relatively when compared with 4, than it is when compared with 2.

227. The general method of expressing the ratio, which one quantity or number bears to another, is by placing two points between them. Thus,

The ratio of	12 to 4 is expressed	by	$12 : 4$.
"	19 to 9 " "	by	$19 : 9$.
"	a to b " "	by	$a : b$.

228. The first term of a ratio is called the *Antecedent*, and the last term the *Consequent.* The antecedents in the preceding ratios are therefore 12, 19, and a; and the consequents, 4, 9, and b.

229. Ratios may therefore be represented in the form of fractions, by making the antecedents the numerators, and the consequents the denominators; thus,

$$\frac{12}{4},\ \frac{19}{9},\ \text{and}\ \frac{a}{b}$$

express the ratios of 12 to 4, of 19 to 9, and of a to b.

230. A ratio is said to be of equality, when the antecedent is equal to the consequent.

Thus the ratio of $12 : 12$, or of $a : a$, is a ratio of equality.

231. A ratio is of greater inequality when the antecedent is greater than the consequent. Thus,

The ratio of $a+b : a$, or of $12 : 6$, is a ratio of greater inequality.

232. A ratio of less inequality is, when the antecedent is less than the consequent. Thus,

The ratio of $a : a+b$, or of $12 : 6$, is a ratio of less inequality.

NOTE. — It is evident, that the ratio of equality may always be represented by unity.

COMPARISON BY RATIOS.

233. If the terms of a ratio are both multiplied or both divided by the same quantity, the value of the ratio is not altered.

The ratio of $a:b$ is expressed by the fraction $\frac{a}{b}$. Let both terms of this fraction be multiplied by n, and it becomes $\frac{na}{nb}$. The ratio of $4:3$ is expressed by the fraction $\frac{4}{3}$; and, if the terms of this fraction be multiplied by 3, it becomes $\frac{12}{9}=\frac{4}{3}$. Now, since the value of a fraction is not altered by multiplying both the numerator and denominator by the same quantity, $\frac{a}{b}=\frac{na}{nb}$, or the ratio $a:b$ is the same as the ratio $na:nb$, and the ratio of $12:9$ is the same as $4:3$. Thus the ratio of $16:12$ by dividing both terms by 4 is the same as $4:3$.

The ratio of $5:7$, by multiplying both terms by 3, is the same as the ratio of $15:21$. And the ratio of $a^2:ab$, by dividing by a, is the same as the ratio of $a:b$.

234. Ratios are compared together by reducing the fractions, which represent them to a common denominator.

Thus the ratios of $7:9$: and $10:13$ are represented by the fractions $\frac{7}{9}$ and $\frac{10}{13}$, which are equivalent to $\frac{91}{117}$ and $\frac{90}{117}$; and since $\frac{91}{117}$ is greater than $\frac{90}{117}$, we infer the ratio of $7:9$ is greater than $10:13$.

235. When the antecedents or consequents are the same in two or more ratios, we immediately compare those ratios together, by expressing them in a fractional form. Thus, since $\frac{17}{5}$ is greater than $\frac{17}{9}$ the ratio, the ratio of $17:5$ is greater than 17 : 9 ; and, since $\frac{a}{a+b}$ is less than $\frac{a}{b}$, the ratio of $a : a+b$ is less than $a : b$.

236. A ratio of greater inequality is diminished, and a ratio of less inequality is increased by adding the same quantity to both terms.

Let $\frac{a}{b}$ represent any ratio, and add n to each of the terms, then these two ratios will be $\frac{a}{b}$ and $\frac{a+n}{b+n}$ which are equivalent to $\frac{ab+an}{b(b+n)}$ and $\frac{ab+bn}{b(b+n)}$. Now, if a be greater than b, $\frac{a}{b}$ is a ratio of greater inequality, and $\frac{ab+an}{b(b+n)}$ is greater than $\frac{ab+bn}{b(b+n)}$; therefore $\frac{a}{b}$ is diminished by adding n to each of the terms. But, if a be less than b, then $\frac{a}{b}$ is a ratio of less inequality, and $\frac{ab+an}{b(b+n)}$ is less than $\frac{ab+bn}{b(b+n)}$; therefore $\frac{a}{b}$ is increased by the addition of n to both terms.

COMPOUND RATIOS.

237. Ratios are compounded by multiplying their antecedents together to form a new antecedent, and their consequents to form a new consequent. The resulting ratio is called the *sum* of the compounding ratios.

Thus, the ratio of $a : b$ is compounded with the ratio of $c : d$ by multiplying the antecedents a and c together for a new antecedent, and the consequents b and d together for a new consequent, and the resulting ratio $ac : bd$ is the sum of the compounding ratios $a : b$ and $c : d$.

If the ratios 4 : 7, 6 : 11, and 7 : 9 are compounded together, the resulting ratio is $4 \times 6 \times 7 : 7 \times 11 \times 9$, or $168 : 693$, which, reduced to its lowest terms by dividing both terms by 21, becomes the ratio $8 : 33$.

238. When any ratio $a : b$ is compounded with itself twice, thrice, or any number of times, denoted by n, then the resulting ratios are $a^2 : b^2$, $a^3 : b^3$, $a^4 : b^4$, &c , are called the duplicate, triplicate, quadruplicate, &c. ratios of the primitive.

As the indices or exponents 2, 3, and n, express the number of times the ratio of $a : b$ is compounded of itself, they are called the *measures* of these ratios.

Since the index may be any quantity, either integral or fractional, let the fraction be $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{m}$, &c., then,

The ratio of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is $\frac{1}{2}$ the ratio of $a : b$.
" " $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ is $\frac{1}{3}$ " " of $a : b$.
" " $a^{\frac{1}{4}} : b^{\frac{1}{4}}$ is $\frac{1}{4}$ " " of $a : b$.
" " $a^{\frac{1}{m}} : b^{\frac{1}{m}}$ is $\frac{1}{m\text{th}}$ " " of $a : a$.

239. The ratios of $a^{\frac{1}{2}} : b^{\frac{1}{2}}$, $a^{\frac{1}{3}} : b^{\frac{1}{3}}$, $a^{\frac{1}{4}} : b^{\frac{1}{4}}$, &c., are also called the subduplicate, subtriplicate, subquadruplicate, &c. ratios of the primitive.

PROPORTION.

240. Proportion consists in the equality of ratios.

Thus, if the ratio of $a : b$ is equal to that of $c : d$, or $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d, are said to be *proportional.* The numbers 3, 12, 4, 16, are proportionals, for $\frac{3}{12} = \frac{1}{4}$, and $\frac{4}{16} = \frac{1}{4}$.

This equality of ratios is expressed by writing the four quantities; thus, $a : b : : c : d$, and read, a is to b as c to d.

241. In algebraic investigations, the quantities are generally expressed like fractions, thus $\frac{a}{b}=\frac{c}{d}$.

In the proportion $a : b :: c : d$, or $\frac{a}{b}=\frac{c}{d}$, a and d are the extremes, and b and c the means. The first term is also called the first antecedent, and the second the first consequent, the third term the second antecedent, and the fourth term the second consequent.

242. If in a series of proportional quantities, each consequent be identical with the next antecedent, these quantities are said to be in continued proportion. Thus $a:b::b:c::c:d::d:e::e:f$, &c., the quantities a, b, c, d, e, f, &c., are said to be in continued proportion.

243. When the second and third terms are identical, as in the proportion $a:b::b:c$, then b is said to be a mean proportional between the extremes a and c; and c is called the third proportional to a and b.

244. The sum of any number of ratios, of which the consequent of the preceding ratio is the antecedent of the succeeding one, is the ratio of the first antecedent to the last consequent.

Let the ratios be $a : b$, $b : c$, $c : d$, $d : e$, $e : f$, then the resulting ratio is $a\times b\times c\times d\times e : b\times c\times d\times e\times f$, or the ratio of $abcde : bcdef$, which, reduced to its least terms by cancelling the same letters in each term, becomes $a : f$, or the first antecedent and the last consequent.

Again; let the ratios be 2 : 3, 3 : 4, 4 : 5, 5 : 7, 7 : 10, then the resulting ratio is,

$$2\times3\times4\times5\times7 : 3\times4\times5\times7\times10, \text{ or } 840 : 4200,$$

which reduced is, 7 : 35, or 1 : 5.

245. Any ratio compounded with a ratio of greater inequality is increased, and compounded with a ratio of less inequality is diminished.

Let $a+b : a$ represent the ratio of greater inequality, and $a : a+b$: : : : of less inequality. Then the ratio of $a+b : a$, compounded with that of $c : d$ gives $ac+bc : ad$, which is evidently greater than the ratio of $c : d$; and the ratio of $a : a+b$ compounded with that of $c : d$ gives $ac : ad+bd$, which is evidently less than the ratio of $c : d$.

Hence, the ratio of $c : d$ is increased by compounding it with the ratio of $a+b : a$, and diminished by compounding it with the ratio of $a : a+bc$.

APPROXIMATION OF RATIOS.

246. The ratio of the powers or roots of two quantities whose difference is small with respect to themselves, is found very nearly by multiplying that difference by the index or exponent of the power or root.

PROPOSITIONS.

Proposition I. If four quantities are proportional, the product of the extremes is equal to the product of the means, and conversely.

Let $a : b : : c : d$, or $\frac{a}{b}=\frac{c}{d}$.

Multiplying both by bd, we obtain $ad=bc$.

Conversely. If the product of any two quantities is equal to the product of any other two, these four quantities are proportional, the factors of either of the product being made the extremes, and the factors of the others the means.

Let $ad=bc$, dividing both by bd, we obtain $\frac{a}{b}=\frac{c}{d}$, or $\frac{c}{d}=\frac{a}{b}$; whence $a : b : : c : d$, or $c : d : : a : b$.

Prop. II. If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean, and conversely.

Let $a : b : : b : c$; $a\times c=b\times b$, or $ac=b^2$.

Conversely. If the product of any two quantities is equal to the square of a third, the third is a mean proportional between the other two.

Let $ac=b^2$, and, dividing both by bc, we obtain $\frac{a}{b}=\frac{b}{c}$, or $a:b::b:c$.

Prop. III. Of four proportionals, any three being given, the fourth may be found.

Let $a:b::c:d$; then $ad=bc$.

Hence, $a=\frac{bc}{d}$; $b=\frac{ad}{c}$; $c=\frac{ad}{b}$; $d=\frac{bc}{a}$.

Hence, of three proportionals, any two being given, the third may be found; for $ad=b^2$, therefore $b=\sqrt{ad}$, $a=\frac{b^2}{d}$, and $d=\frac{b^2}{a}$.

Prop. IV. Quantities, which have the same ratio to the same quantity, are equal to one another, and conversely.

Let $a:b::c:b$, then $\frac{a}{b}=\frac{c}{b}$; and multiplying each by b, we obtain $a=c$.

Conversely. Quantities, which are equal to one another, have the same ratio to the same quantity.

Let $a=c$, and let b be a third quantity; then, dividing both by b, we obtain

$$\frac{a}{b}=\frac{c}{b}, \text{ therefore } a:b::c:b.$$

Prop. V. Ratios that are equal to the same ratio, are equal to one another.

Let $a:b::e:f$, and $c:d::e:f$; then, also, $a:b::c:d$.

Since $\frac{a}{b}=\frac{e}{f}$, and $\frac{c}{d}=\frac{e}{f}$, it is evident $\frac{a}{b}=\frac{c}{d}$, and therefore $a:b::c:d$.

Or, let $2:4::8:16$, and $3:6::8:16$,

Then $2:4::3:6$; for $\frac{2}{4}=\frac{1}{2}$, and $\frac{3}{6}=\frac{1}{2}$.

Prop. VI. If four quantities are proportionals, they will also be proportionals when taken *inversely;* that is, the second will have the same ratio to the first that the fourth has to the third.

Let $a:b::c:d$, then $b:a::d:c$.

Since by Prop. I. $bc=ad$,

And dividing by ac, we obtain $\frac{b}{a}=\frac{d}{c}$.

Hence, $b:a::d:c$.

Prop. VII. If four quantities are proportionals, they shall also be proportionals when taken alternately, that is, the first will have the same ratio to the third, that the second has to the fourth.

Let $a:b::c:d$; then, also, $a:c::b:d$;

As $\frac{a}{b}=\frac{c}{d}$, if we multiply each quantity by $\frac{b}{c}$, we obtain $\frac{ab}{bc}=\frac{cb}{cd}$; which, reduced, is $\frac{a}{c}=\frac{b}{d}$, therefore $a:c::b:d$.

This may be illustrated by numbers; thus,

Let $2:4::3:6$, then $2:3::4:6$;

As $\frac{2}{4}=\frac{3}{6}$, if we multiply each side of the equation by $\frac{4}{3}$ the result will be $\frac{2}{4}\times\frac{4}{3}=\frac{3}{6}\times\frac{4}{3}$; $\frac{8}{12}=\frac{12}{18}$; $\frac{2}{3}=\frac{4}{6}$; therefore $2:3::4:6$.

Prop. VIII. If four quantities are proportionals, they will also be proportionals when taken jointly, that is, the sum of the first and second will have the same ratio to the second, that the sum of the third and fourth has to the fourth.

Let $a : b :: c : d$, then $a+b : b :: c+d : d$;

Since $\frac{a}{b}=\frac{c}{d}$, we add 1 to each quantity, and obtain $\frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$, therefore $a+b : b :: c+d : d$.

This, also, may be made evident by taking numbers; thus,

Let $2 : 4 :: 3 : 6$, then $2+4 : 4 :: 3+6 : 6$.

Since $\frac{2}{4}=\frac{3}{6}$, we add 1 to each number, and obtain

$$\frac{2}{4}+1=\frac{3}{6}+1, \text{ or } \frac{2+4}{4}=\frac{3+6}{6},$$

Therefore, $2+4 : 4 :: 3+6 : 6$.

Prop. IX. If four quantities are proportionals, they will also be proportionals by *separation;* that is, the difference between the first and second will have the same ratio to the second, that the difference between the third and fourth has to the fourth.

Let $a : b :: c : d$, then $a-b : b :: c-d : d$.

Since $\frac{a}{b}=\frac{c}{d}$, we will subtract 1 from each quantity and we obtain $\frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b} : \frac{c-d}{d}$,

Therefore, $a-b : b :: c-d : d$.

This demonstration may be illustrated by numbers; thus,

Let $4 : 2 :: 6 : 3$, then $4-2 : 2 :: 6-3 : 3$.

Since $\frac{4}{2}=\frac{6}{3}$ we will subtract 1 from each term, and we have

$$\frac{4}{2}-1=\frac{6}{3}-1, \text{ or } \frac{4-2}{2}=\frac{6-3}{3},$$

Therefore, $4-2 : 2 :: 6-3 : 3$.

Prop. X. If four quantities are proportionals, they will also be proportionals by *conversion*, that is, the first term will have the same ratio to the sum or difference of the first and second, that the third has to the sum or difference of the third and fourth.

Let $a : b :: c : d$; then $a : a \pm b : c :: c \pm d$. Since $\frac{a}{b} = \frac{c}{d}$, and by Prop. VIII. and IX. $\frac{a \pm b}{b} = \frac{c \pm d}{d}$, invert these fractions, and we have $\frac{b}{a \pm b} = \frac{d}{c \pm d}$; and by multiplying the one by $\frac{a}{b}$, and the other by its equal $\frac{c}{d}$, we obtain $\frac{b}{a \pm b} \times \frac{a}{b} = \frac{d}{c \pm d} \times \frac{c}{d}$, or $\frac{a}{a \pm b} = \frac{c}{c \pm d}$, therefore $a : a \pm b :: c : c \pm d$.

Let the pupil prove this by numbers.

PROP. XI. If four quantities are proportionals, the sum of the first and second has the same ratio to their difference, that the sum of the third and fourth has to their difference.

Let $a : b :: c : d$; then, also, $a + b : a - b :: c + d : c - d$.

For by Prop. VIII. and IX. by alternation, $a + b : c + d :: b : d$; and $a - b : c - d :: b : d$; hence by Prop. V. $a + b : c + d :: a - b : c - d$, and, by alternation, $a + b : a - b :: c + d : c - d$.

This is illustrated by numbers, thus, let $8 : 6 :: 12 : 9$; then $8 + 6 : 8 - 6 :: 12 + 9 : 12 - 9$.

For taking Prop. VIII. and IX. by alternation, $8 + 6 : 12 + 9 :: 6 : 9$; and by Prop. V. $8 + 6 : 12 + 9 :: 8 - 6 : 12 - 9$; therefore, by alternation, $8 + 6 : 8 - 6 :: 12 + 9 : 12 - 9$.

PROP. XII. In any number of proportionals, any antecedent has the same ratio to its consequent, that the sum of all the antecedents has to the sum of all its consequents.

Let $a : b :: c : d :: e : f :: g : h$; then, also, $a : b :: a + c + e + g : b + d + f + h$.

Since $ab = ba$, $ad = bc$, $af = be$, $ah = bg$, we have

$$a(b+d+f+h) = b(a+c+e+g);$$

Whence, $\frac{a}{b}=\frac{a+c+e+g}{b+d+f+h}$;

Therefore, $a : b :: a+c+e+g : b+d+f+h$.

In like manner it may be shown, that $c : d :: a+c+e+g : b+d+f+h$.

This proposition may be illustrated by numbers, thus,

Let $2 : 3 :: 4 : 6 :: 8 : 12 :: 14 : 21$;

Then $2 : 3 :: 2+4+8+14 : 3+6+12+21 = 2 : 3 :: 28 : 42$.

Prop. XIII. In two or more ranks of proportionals, the product of the correspondent terms are also proportionals.

$$\text{Let}\left.\begin{array}{r} a : b :: c : d, \\ e : f :: g : h, \\ i : k :: l : m. \end{array}\right\}\text{Then, also, } aei : bfk : cgl : dhm.$$

DEMONSTRATION.

Since $\frac{a}{b}=\frac{c}{d}, \frac{e}{f}=\frac{g}{h}, \frac{i}{k}=\frac{l}{m}; \quad \frac{a\times e\times i}{b\times f\times k}=\frac{c\times g\times l}{d\times h\times m}$;

Whence $\frac{aei}{bfk}=\frac{cgl}{dhm}$, therefore $aei : bfk :: cgl : dhm$.

Q. E. D.

ILLUSTRATION BY NUMBERS.

Let $2 : 3 :: 4 : 6$

$4 : 5 :: 8 : 10$

$6 : 7 :: 12 : 14$

Then $2\times4\times6 : 3\times5\times7 :: 4\times8\times12\times : 6\times10\times14$,

Whence, $48 : 105 :: 384 : 840$.

Prop. XIV. If there are any number of quantities more than two, and as many others, which, taken two and two in order, are proportionals; then, by equality, are the extreme terms in the former series proportional to the extreme terms in the latter.

Let a, b, c, d, be any number of quantities, and let e, f, g, h, as many others.

$$\left.\begin{array}{l} \text{Let } a:b::e:f, \\ \quad b:c::f:g, \\ \quad c:d::g:h. \end{array}\right\} \text{Then, also, } a:d::e:h.$$

DEMONSTRATION.

Since $\frac{a}{b}=\frac{e}{f}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{g}{h}$, we obtain by multiplying the alternate fractions together, $\frac{abc}{bcd}=\frac{efg}{fgh}$, or $\frac{a}{d}=\frac{e}{h}$; therefore $a:d::e:h$.

ILLUSTRATION BY NUMBERS.

$$\left.\begin{array}{l} \text{Let } 2:\ 3::4:\ 6 \\ \quad 3:\ 4::6:\ 8 \\ \quad 4:12::8:24 \end{array}\right\} \text{Then, } 2:12::4:24.$$

By multiplying the alternate fractions, we have

$$2\times3\times4:3\times4\times12::4\times6\times8:6\times8\times24.$$

Whence, 24 : 144 : : 192 : 1152 or 2 : 12 : : 4 : 24.

PROP. XV. If there are any number of quantities more than two, and as many others which taken two and two, in cross order, are proportionals; then inversely, by equality, are the extreme terms in the first rank proportional to the extreme terms in the second.

Let a, b, c, d, be any number of terms, and e, f, g, h, as many others, and

$$\left.\begin{array}{l} \text{Let} \quad a:b::g:h \\ \quad b:c::f:g \\ \quad c:d::e:f \end{array}\right\} \text{Then, also, } a:d::e:h.$$

DEMONSTRATION.

Since $\frac{a}{b}=\frac{g}{h}$, $\frac{b}{c}=\frac{f}{g}$, and $\frac{c}{d}=\frac{e}{f}$; by multiplying the alternate fractions together, we obtain

$$\frac{abc}{bcd}=\frac{gfe}{hgf}, \text{ or } \frac{a}{d}=\frac{e}{h},$$

Therefore, $a:d::e:h$.

ILLUSTRATION BY NUMBERS.

$$\text{Let}\quad \left.\begin{array}{r} 2 : 3 : : 8 : 12 \\ 3 : 4 : : 6 : 8 \\ 4 : 3 : : 8 : 6 \end{array}\right\} \text{Then, } 2 : 3 : : 8 : 12.$$

$$2\times3\times4 : 3\times4\times3 : : 8\times6\times8 : 12\times8\times6,$$

Whence, $24 : 36 : : 384 : 576.$

By dividing the first two terms by 12, and the last two by 48, we obtain $2 : 3 : : 8 : 12.$

PROP. XVI. When four quantities are proportionals, if the first and second are multiplied, or divided by the same quantity, and also the third and fourth by the same quantity, the resulting quantities will be proportionals.

Let $a : b : : c : d$; then, also, $ma : mb : nc : nd$.

DEMONSTRATION.

Since $\frac{a}{b}=\frac{c}{d}$, we multiply both terms of the first by m, and both terms of the last by n, and we obtain $\frac{ma}{mb}=\frac{nc}{nd}$;

Therefore, $ma : mb : : nc : nd,$

where m and n may be any quantities, either integral or fractional.

ILLUSTRATION BY NUMBERS.

Let $2 : 4 : : 3 : 6$. Now, if we multiply the first two numbers by 7, and the last two numbers by 9, their products will be proportionals. Thus,

$$2\times7 : 4\times7 : : 3\times9 : 6\times9=14 : 28 : : 27 : 54;$$

and, if any other numbers were taken instead of 7 and 9, the products would be proportionals.

PROP. XVII. When four quantities are proportionals, if the first and third are multiplied or divided by the same quantity, and also the second and fourth by the same quantity, the resulting quantities will be proportionals.

Let $a : b : : c : d$, then, also, $ma : nb : : mc : nd$.

DEMONSTRATION.

Since $\frac{a}{b}=\frac{c}{d}$, multiply both these quantities by $\frac{m}{n}$, and we obtain $\frac{ma}{nb}=\frac{mc}{nd}$, therefore $ma : nb :: mc : nd$, where m and n may be any quantities, either integral or fractional.

ILLUSTRATION BY NUMBERS.

Let 12 : 4 : : 18 : 6, and we will multiply the first and third by 2, and the second and fourth terms by 4.

Thus, $12\times2 : 4\times4 :: 18\times2 : 6\times4=24 : 16 :: 36 : 24.$

It is evident these terms are proportionals;

For $$\frac{24}{16}=\frac{36}{24} \text{ or } \frac{12}{8}=\frac{12}{8}.$$

And, if we divide the first and third terms by 3, and the second and fourth terms by 2, their quotients will be proportionals.

Thus, $$12\div3 : 4\div2 :: 18\div3 : 6\div2,$$

Or, $$4 : 2 :: 6 : 3,$$

Whence, $$\frac{4}{2}=\frac{6}{3}.$$

If any other numbers be taken for multiplying or dividing, the result will be the same.

PROP. XVIII. If four quantities are proportionals, the like powers or roots of these quantities are also proportionals.

Let $a : b :: c : d$; then also $a^m : b^m :: c^m : d^m$.

Since $\frac{a}{b}=\frac{c}{d}$, raise each of these fractions to the power expressed by m; then $\left(\frac{a}{b}\right)^m=\left(\frac{c}{d}\right)^m$ or $\frac{a^m}{b^m}=\frac{c^m}{d^m}$, therefore $a^m : b^m :: c^m : d^m$, where m may be any quantity either integral or fractional.

ILLUSTRATION.

Let $2 : 3 :: 4 : 6$, then $2^3 : 3^3 :: 4^3 : 6^3$. If we raise each of these terms to the third power, the result will be

$$2\times2\times2=8 : 3\times3\times3=27 :: 4\times4\times4=64 : 6\times6\times6=216.$$

That 8, 27, 64, and 216, are proportionals, is evident from the fact, that $\frac{8}{27}=\frac{64}{216}$, and being reduced to their lowest terms $\frac{8}{27}=\frac{8}{27}$.

Prop. XIX. Of any number of quantities in continued proportion, the first has to the third the duplicate ratio, to the fourth the triplicate ratio, to the fifth the quadruplicate ratio, &c. of that which it has to the second, or of that which the second has to the third, &c.

Let $a : b :: b : c :: c : d :: d : e :: e : f ::$ &c., &c.
Then $a : c :: a^2 : b^2$, or in the duplicate ratio of $a : b$.
$a : d :: a^3 : b^3$, or in the triplicate ratio of $a : b$.
$a : e :: a^4 : b^4$, or in the quadruplicate ratio of $a : b$.

DEMONSTRATION.

1st. $a : b :: b : c$, or by Prop. XVIII. $a^2 : b^2 :: b^2 : c^2$; but by Prop. II. $b^2=ac$, therefore $a^2 : b^2 :: ac : c^2$, or $a^2 : b^2 :: a : c$, hence $a : c :: a^2 : b^2$; also $a^2 : ac :: b^2 : c^2$; therefore $a : c :: b^2 : c^2$.

2d. $a : c :: a^2 : b^2$; but $c : d :: a : b$; therefore
$$a : d :: a^3 : b^3 :: b^3 : c^3 :: c^3 : d^3.$$

3d. But $d : e :: a^3 : b^3$; therefore
$$a : e :: a^4 : b^4 :: b^4 : c^4 :: c^4 : d^4 :: d^4 : e^4.$$

The above may be easily illustrated by numbers.

PROBLEMS FOR PROPORTION.

1. Divide 50 into two such parts, that the greater increased by 3 shall be to the less diminished by 3 as 3 to 2.

Let $x=$ the greater number, and $50-x$ the less.

Then $x+3 : 50-x-3 :: 3 : 2$

Multiplying extremes, $2x+6=150-3x-9$

Transposing, $5x=135$

Dividing, $x=27$ the greater,

And $50-27=23$ the less.

2. What number is that, to which if 3, 8, 12, and 20, be severally added, their sums shall be proportional?

Let $x=$ the number,

Then, $x+3 : x+8 :: x+12 : x+20.$

Multiplying extremes, $x^2+23x+60=x^2+20x+96.$

$23x-20x=96-60.$

$x=12.$ *Ans.*

VERIFICATION.

$12+3 : 12+8 :: 12+12 : 12+20=15 : 20 :: 24 : 32.$

3. If Mars when in opposition to the sun is 49,000,000 miles from the earth, and the quantity of matter in the earth is 11 times greater than that in Mars, at what distance from the earth in a direction towards Mars will a body remain at rest? See Art. 213.

Let $x=$ the distance from the earth,

Then $49,000,000-x=$ the distance from Mars,

And let $a=49,000,000$,

Then, $x^2 : (a-x)^2 :: 1 : 11$

Multiplying extremes, $11x^2=a^2-2ax+x^2$

Transposing, $10x^2+2ax=a^2$

Reducing, $x^2+\frac{ax}{5}=\frac{a^2}{10}$

Completing the squares, $x^2+\frac{ax}{5}+\frac{a^2}{100}=\frac{a^2}{10}+\frac{a^2}{100}=\frac{11a^2}{100}$

Evolving, $x+\frac{a}{10}=\frac{1}{10}\sqrt{11a^2}$

Transposing, $x=\frac{1}{10}\sqrt{11a^2}-\frac{a}{10}$

And by supplying the value of a we have

$$x=\frac{1}{10}\sqrt{\left(11(49,000,000)^2\right)}-\frac{49,000,000}{10}=11,351,430 \text{ miles.}$$

Ans.

4. There are two numbers which are to each other as 5 to 3, and if 4 be added to the greater and 8 to the less, they will then be to each other as 6 to 5. What are the numbers?

Ans. 20 and 12.

5. Divide the number 60 into two such parts that their product shall be to the difference of their squares as 2 to 3.

Ans. 40 and 20.

6. I have two square house lots, which, together, contain 208 square rods; and the area of the greater is to the area of the less as 9 to 4. How many more square rods are there in the greater than in the less? *Ans.* 80 square rods.

7. The product of two numbers is 12, and the difference of their cubes is to the cubes of their difference as 13 to 4. What are the numbers? *Ans.* 2 and 6.

8. Divide the number 100 into two such parts that 6 times their product shall be to the sum of their squares as 24 to 17. What are those parts? *Ans.* 80 and 20.

9. There are two numbers, whose product is 35, and the difference of their squares is to the square of their difference as 6 to 1. What are the numbers? *Ans.* 7 and 5.

10. There are two numbers in the triplicate ratio of 4 to 1, whose mean proportion is 32. What are the numbers?

Ans. 256 and 4.

11. Divide 20 into two such numbers, that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 9 to 4. What are those parts?

Ans. 12 and 8.

12. Divide 26 into three such parts, that the first shall have the same ratio to the second, that the second has to the third.

Ans. 2, 6, and 18.

SECTION XVIII.

ARITHMETICAL PROGRESSION.

ART. **247.** An Arithmetical Progression is a regular increase or decrease of numbers or quantities by a constant difference.

It is sometimes called *Progression* by *Difference.*

248. The constant difference is called the *Common Difference*, or *ratio* of the progression.

Thus, let there be the two following series.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
First series,	1,	4,	7,	10,	13,	16,	19,	22=92.
Second series,	30,	26,	22,	18,	14,	10,	6,	2=128.

249. The numbers, which form the series, are called the *terms* of the progression.

250. The first is called an *ascending series* of progression, where the first term is 1, the common difference 3, the number of terms 8, the last term 22, and the sum of the series 92.

251. The second is called a *descending series* of progression, where the first term is 30, the common difference —4, the number of terms 8, the last term 2, and the sum of the series 128.

252. The first and last terms of the progression are called *extremes*, and the other terms are the *means.*

253. The number of common differences in any number of terms is *one less* than the number of terms.

Hence, if there be 8 terms, the number of common differences will be 7, and the sum of their differences will be equal to the difference of the extremes.

We therefore infer, that, if the difference of the extremes be added to the first term, the sum will be the last term; also, if the difference of the extremes be taken from the last term, the remainder will be the first term.

254. Also, if the sum of the common differences be divided by the number of common differences, the quotient will be the common difference.

To illustrate this, we will examine the following series.

(1) (2) (3) (4) (5) (6) (7)
2, 5, 8, 11, 14, 17, 20.

Here, the first term is 2, last term 20, the number of terms 7, and the common difference 3.

Now, if we had only the first term, number of terms, and common difference to find the last term, we should have only to add the difference of the extremes to the first term.

The common difference is 3, and as there are 7 terms, the number of common differences is 6. The difference of the extremes will therefore be $6\times3=18$, and the last term will be $2+18=20$.

Hence, having the first term, common difference, and number of terms given to find the last term, we have the following

RULE. *Multiply the number of terms,* less one, *by the common difference, and to the product add the first term.*

Again, if we invert the terms, we have

(1) (2) (3) (4) (5) (6) (7)
20, 17, 14, 11, 8, 5, 2.

Here we have 20 for the first term, -3 for the common difference, and 7 for the number of terms, to find the last term.

$6\times-3=-18$; $20-18=2$ the last term.

The pupil will perceive that 18 is a negative term; and to add a negative term to a positive is to write their difference.

Again, we have given the extremes 2 and 20, and number of terms 7, to find the common difference.

Here the number of common differences is 6; for we have before shown, that the number of common differences is always *one less* than the number of terms; therefore, $18\div6=3$ the common difference.

255. The principles of an arithmetical progression may be well illustrated by *literal terms.*

Let a be the first term of an ascending series, and d the common difference; then the second term will be $a+d$, and the third term $a+2d$, and the series will be

$$\begin{array}{cccccc} (1) & (2) & (3) & (4) & (5) & (6) \\ a, & a+d, & a+2d, & a+3d, & a+4d, & a+5d. \end{array}$$

If it be required to form a descending series, when the first term is a and the common difference $-d$, it will be thus,

$$\begin{array}{cccccc} (1) & (2) & (3) & (4) & (5) & (6) \\ a, & a-d, & a-2d, & a-3d, & a-4d, & a-5d. \end{array}$$

256. It is evident, that the last term in both series is equal to the first term with the common difference repeated as many times, *wanting one*, as there are terms in the series.

Hence, if n represent the number of terms, the following will be the formula to find L, the last term.

$$L=a+(n-1)d.$$

EXAMPLES.

1. If the first term be 7, the common difference 4, and the number of terms 20, required the last term.

$$L=a+(n-1)d=7+(20-1)4=83. \quad \textit{Ans.}$$

2. If the first term be 3, the common difference 5, required the 50th term.

$$L=a+(n-1)d=3+(50-1)5=248. \quad \textit{Ans.}$$

3. If the first term be 90, the common difference -7, required the 10th term.

$$L=a+(n-1)(-d)=90+(10-1)(-7)=27.\ \textit{Ans.}$$

4. If the first term be $\frac{3}{4}$, the common difference $1\frac{1}{3}$, what is the 20th term?

$$L=a+(n-1)d=\tfrac{3}{4}+(20-1)1\tfrac{1}{3}=26\tfrac{1}{12}.\ \textit{Ans.}$$

5. If the first term be 18, the common difference -4, what is the 10th term?

$$L=a+(n-1)(-d)=18+(10-1)(-4)=-18.\ \textit{Ans.}$$

257. The formula for obtaining the first term, a, is obtained from the former by transposition.

Thus, if $L=a+(n-1)d$, then, by transposition,

$$a=L-(n-1)d.$$

6. If the last term is 25, the number of terms 6, and the common difference 2, required the first term.

$$a=L-(n-1)d=25-(6-1)2=15.\ \textit{Ans.}$$

7. If the last term is 50, the common difference 6, the number of terms 10, required the first term.

$$a=L-(n-1)d=50-(10-1)6=-4.\ \textit{Ans.}$$

8. If the last term is 27, the common difference $2\frac{1}{2}$, number of terms 10, required the first term.

$$a=L-(n-1)d=27-(10-1)2\tfrac{1}{2}=4\tfrac{1}{2}.\ \textit{Ans.}$$

258. The formula for obtaining the common difference, d, is obtained from the first by transposition and division.

Thus, $L=a+(n-1)d.$

Then, by transposition, $L-a=(n-1)d.$

And by division, $\dfrac{L-a}{n-1}=d.$

Changing terms, $d=\dfrac{L-a}{n-1}.$

9. If the extremes are 6 and 30, and the number of terms 13, what is the common difference?

$$d=\frac{L-a}{n-1}=\frac{30-6}{13-1}=2. \quad \textit{Ans.}$$

10. If the extremes are $\frac{3}{4}$ and $15\frac{3}{4}$, and the number of terms 11, what is the common difference?

$$d=\frac{L-a}{n-1}=\frac{15\frac{3}{4}-\frac{3}{4}}{11-1}=1\tfrac{1}{2}. \quad \textit{Ans.}$$

259. The formula for obtaining the number of terms may be obtained from the first formula.

Thus, $L=a+(n-1)d.$

By transposition, $L-a=(n-1)d.$

By division, $\frac{L-a}{d}=n-1.$

By transposition, $\frac{L-a}{d}+1=n.$

11. If the extremes are 3 and 39, and the common difference 2, what is the number of terms?

$$n=\frac{L-a}{d}+1=\frac{39-3}{2}+1=19. \quad \textit{Ans.}$$

12. If the first term be 5, the last term 89, the common difference 7, required the number of terms?

$$n=\frac{L-a}{d}+1=\frac{89-5}{7}+1=13. \quad \textit{Ans.}$$

Having, therefore, any three of the four terms given, the other may be found, as we have demonstrated above, by the following

FORMULÆ.

(1.) To find the last term.

$$L=a+(n-1)d.$$

(2.) To find the first term.

$$a=L-(n-1)d.$$

(3.) To find the common difference.

$$d=\frac{L-a}{n-1}.$$

(4.) To find the number of terms.

$$n=\frac{L-a}{d}+1.$$

When the series are descending, the unknown difference is a minus quantity in the 1st and 2d formulæ; thus $-d$.

13. A man travelled 10 days; the first day he went 8 miles, the second day 13 miles, and thus increased his distance each day 5 miles. How far did he travel the last day?

Ans. 53 miles.

14. John Smith's family expenses for the first year were $500. But, after he had been married 12 years, he found his last year's expenses to have been $1325. By how much did he increase his expenses yearly? *Ans.* $75.

15. A man set out from Boston to travel into the country. The first day he travelled 12 miles, the second day 9 miles, the third day 6 miles, and thus continuing to travel each day 3 miles less than the preceding. How far did he go the tenth day? *Ans.* —15 miles.

260. To find the sum of the series.

ARITHMETICAL SERIES.

	(1)	(2)	(3)	(4)	(5)	(6)	
Let	2,	5,	8,	11,	14,	17,	be the series.
And	17,	14,	11,	8,	5,	2,	same series inverted.
	19,	19,	19,	19,	19,	19,	sum of both series.

LITERAL SERIES.

	(1)	(2)	(3)	(4)	(5)	(6)	
Let	a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$,	$a+5d$	be a series.
And	$a+5d$,	$a+4d$,	$a+3d$,	$a+2d$,	$a+d$,	a	same series inverted.

$2a+5d$, $2a+5d$, $2a+5d$, $2a+5d$, $2a+5d$, $2a+5d$, sum of both series.

We perceive from the above arithmetical and literal series, that the sum of the extremes is equal to the sum of any of the means equally distant from either extreme; and that, by adding the two series in their present arrangement, we have the same number for the same successive terms; also, that the sum of both series is twice the sum of either series. Therefore, if 19, the sum of the extremes in the arithmetical series, be multiplied by 6, the number of terms, the product will be the sum of both series. Thus, $19\times6=114$ sum of both series. Therefore $114\div2=57$ will be the sum of either series.

Again, $2a+5d$ is the sum of the extremes in the *literal* series; and, if this sum be multiplied by 6, the number of terms, the product will be the sum of both series. Thus, $(2a+5d)\,6=12a+30d$, sum of both series. And $(12a+30d)\div2=6a+15d$ the sum of either series.

Therefore, in all cases, we find that the sum of the series is equal to the sum of the extremes multiplied by half the number of terms; or, the number of terms multiplied by half the sum of the extremes.

If, therefore, the sum of any series be denoted by S, the following will be the formula for obtaining its value.

$$S=\left(\frac{L+a}{2}\right)n.$$

Therefore, if the extremes and the number of terms are given to find the sum of the series, we adopt the following

RULE. *Multiply half the sum of the extremes by the number of terms.*

The two following formulæ, or equations, contain five quantities; a, the first term of a progression; L, the last term; d, the common difference; n, the number of terms; and S, the sum of the series.

If any three of these be given, the other two may be obtained.

1. $$L=a+(n-1)d.$$

2. $$S=\left(\frac{L+a}{2}\right)n.$$

261. The pupil will find, that twenty different cases may arise, which may be solved by different combinations of the above equations.

To find n in the last equation.

$$\left(S=\frac{L+a}{2}\right)n.$$

By multiplication, $2S=(L+a)n.$

By division, $\dfrac{2S}{L+a}=n.$

Therefore, $n=\dfrac{2S}{L+a}.$

If therefore the extremes and the sum of the series be given to find the number of terms, we divide twice the sum of the series by the sum of the extremes.

16. Let the extremes be 3 and 39, and the sum of the series 399, to find the number of terms.

$$n=\frac{2S}{L+a}=\frac{2\times 399}{39+3}=19. \quad \textit{Ans.}$$

262. To find the least term, L, from the second equation.

$$S=\left(\frac{L+a}{2}\right)n.$$

By multiplication, $2S=(L+a)n.$

By division, $\dfrac{2S}{n}=L+a.$

By transposition, $\dfrac{2S}{n}-a=L.$

By transposition of terms, $L=\dfrac{2S}{n}-a.$

Therefore having the first term, number of terms, and sum of the series given to find the last term, we divide twice the sum of the series by the number of terms, and subtract the first term from the quotient.

263. To find the first term, a, from the second equation.

$$S=\left(\frac{L+a}{2}\right)n.$$

Multiplying, $2S=(L+a)n.$

Dividing, $\frac{2S}{n}=L+a.$

Transposing, $\frac{2S}{n}-L=a.$

Changing terms, $a=\frac{2S}{n}-L.$

Therefore, having the last term, number of terms, and sum of the series, given to find the first term, we divide twice the sum of the series by the number of terms, and subtract the last term from the quotient.

17. Let the last term be 39, number of terms 19, and the sum of the series 399, to find the first term.

$$a=\frac{2S}{n}-L=\frac{2\times399}{19}-39=3.\ \textit{Ans.}$$

264. To find the common difference, d, from the 1st and 2d equation.

We find the value of L in the 1st equation to be

$$L=a+(n-1)d.$$

Substituting this value of L for S in the 2d equation, and then transposing, we have

$$d=\frac{2S-2an}{n(n-1)}.$$

18. If the first term be 5, the number of terms 15, and the sum of the series 285, what is the common difference?

Ans. 2.

19. If the first term be 3, the number of terms 19, and the sum of the series 399, what is the common difference ?

Ans. 3.

20. If the first term be 7, the number of terms 8, and the sum of the series 100, what is the common difference ?

Ans.

PROBLEMS.

1. The first term is 5, the common difference 3. What is the 7th term ? *Ans.*

2. The first term is 3, the common difference $4\frac{1}{3}$. What is the 5th term ? *Ans.*

3. The first term is 18, the common difference $\frac{1}{4}$. What is the 7th term ? *Ans.*

4. The first term is 7, the common difference $2\frac{1}{2}$, and the number of terms 5. Required the last term. *Ans.*

5. The first term is $\frac{3}{4}$, the common difference $\frac{4}{5}$. What is the 10th term ? *Ans.*

6. The first term is 0, the common difference $1\frac{1}{2}$. What is the 20th term ? *Ans.*

7. The first term is 10, the common difference -2. What is the 4th term ? *Ans.*

8. The first term is -8, the common difference -3. What is the 10th term ? *Ans.*

9. The first term of a descending series is 85, common difference 7. Required the 10th term. *Ans.*

10. The first term is $3\frac{1}{3}$, the common difference $2\frac{1}{4}$. What is the 5th term, and the sum of the series ? *Ans.*

11. The first term in a descending series is $2\frac{1}{2}$, the common difference is $\frac{1}{4}$. What is the 10th term and the sum of the series ? *Ans.*

12. The first term is a, the common difference is d. What is the nth term? *Ans.*

13. What is the sum of the odd numbers from 1 to 100? *Ans.*

14. If the first term is $4\frac{1}{2}$, the common difference $3\frac{1}{2}$, and number of terms 8, what is the sum of the series? *Ans.*

15. If the first term is 7, the common difference -4, and the number of terms 6, what is the sum of the series? *Ans.*

16. If the first term is 5, the last term 19, and the number of terms 6, what are the other terms of the progression? *Ans.*

17. If the extremes are -9 and 18, and the number of terms 5, what are the other terms of the progression? *Ans.*

18. If the last term of an ascending series is 20, common difference 5, and the number of terms 8, what is the sum of the series? *Ans.*

19. There is a number consisting of three digits in arithmetical progression, whose sum is 12; and, if 396 be added to the number, the digits will be inverted. What is the number? *Ans.*

20. There is a certain island 50 miles in circumference. Two men, A and B, set out to travel round it. A goes 10 miles each day. B goes 2 miles the first day, 5 miles the second day, and 8 miles the third day, travelling each day 3 miles farther than the day preceding. How far will A and B be apart the 8th day? *Ans.* 30 miles.

21. John Smith and John Jones set out from Boston for the city of Washington, the distance being 440 miles. Smith started 5 days before Jones, and travels 15 miles per day. Jones

travels 25 miles the first day, 23 miles the second day, and 21 miles the third day, travelling each day 2 miles less than the preceding. What will be the difference between Smith and Jones at the end of the 20th day, and how far will each be from Washington? *Ans.*

22. If the first term be $\frac{1}{2}$, the common difference $-\frac{1}{6}$ and the number of terms 20, what is the last term and the sum of the series?

Ans. $\left\{ \begin{array}{l} \text{Last term } -2\frac{2}{3}. \\ \text{Sum of the series } -21\frac{2}{3}. \end{array} \right.$

23. If one extreme be $\frac{1}{3}$, the common difference $-\frac{1}{12}$, and the sum of the series $-1\frac{1}{2}$, what is the number of terms? *Ans.* 12.

24. If the first term be $\frac{7}{12}$, last term $2\frac{1}{2}$, and the sum of the series 37, what is the number of terms? *Ans.* 24.

25. If the first term be 3, the last term 17, and the number of terms 29, what are the terms of the series?
Ans. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, &c.

26. The sum of the series is $16\frac{1}{4}$, the number of terms 10, and the common difference $\frac{1}{4}$, to find the first term.
Ans. $\frac{1}{2}$.

27. The first term of an arithmetical series is -5, the common difference $1\frac{1}{2}$, what is the 9th term? *Ans.* 7.

28. What are the three means between -1 and 15?
Ans. 3, 7, and 11.

29. The first term is $1\frac{1}{4}$, number of terms 10, and the sum of the series $6\frac{7}{8}$. What is the common difference?
Ans. $-\frac{1}{8}$.

30. There are three numbers in arithmetical progression, whose sum is 10, and the product of the second and third is $33\frac{1}{3}$. What are those numbers? *Ans.* $-3\frac{1}{3}$, $3\frac{1}{3}$, and 10.

31. The number of terms of an arithmetical progression is equal to $\frac{1}{2}$ the common difference, the last term is equal to 4 times the first, and the sum of the series is equal to $\frac{3}{4}$ the square of the first term. What are the series, and the sum of the series?

Ans. $\left\{\begin{array}{l}\text{The series 20, 32, 44, 56, 68, 80.}\\ \text{Sum of the series, 300.}\end{array}\right.$

32. There are four numbers in arithmetical progression, whose sum is 28, and the sum of whose squares is 216. What are those numbers? *Ans.* 4, 6, 8, and 10.

33. Find three numbers in arithmetical progression, whose sum is 9, and the sum of whose cubes is 99.

Ans. 2, 3, and 4.

34. What are those four numbers in arithmetical progression, the sum of whose squares of the first two terms is 34, and the sum of the squares of the last two is 130?

Ans. 3, 5, 7, and 9.

35. A certain number consists of three digits, which are in arithmetical progression; and, if the number be divided by the sum of its digits, the quotient will be $27\frac{4}{7}$; but, if 396 be added to the number, the digits will be inverted. Required the number. *Ans.* 579.

36. What are those four numbers in arithmetical progression, the sum of the squares of whose extremes is 90, and the sum of the squares of the means is 74?

Ans. 3, 5, 7, and 9.

37. What are those four numbers in arithmetical progression, whose sum is 14, and whose continued product is 120?

Ans. 2, 3, 4, and 5.

38. There are 4 numbers in arithmetical progression, the product of whose extremes is 112, and that of the means 120. What are the numbers? *Ans.* 8, 10, 12, and 14.

There are twenty different cases in Arithmetical Progression, all of which are exhibited in the following TABLE.

No.	Given.	Requir'd	Formulæ.
1	a, d, n	l	$l=a+(n-1)d.$
2	a, d, S		$l=-\frac{1}{2}d\pm\sqrt{2dS+(a-\frac{1}{2}d)^2}.$
3	a, n, S		$l=\frac{2S}{n}-a.$
4	d, n, S		$l=\frac{S}{n}+\frac{(n-1)d}{2}.$
5	a, d, n	S	$S=\frac{1}{2}n[2a+(n-1)d].$
6	a, d, l		$S=\frac{l+a}{2}+\frac{l^2-a^2}{2d}.$
7	a, l, n		$S=\left(\frac{l+a}{2}\right)n.$
8	d, n, l		$S=\frac{1}{2}n(2l-(n-1)d).$
9	a, n, l	d	$d=\frac{l-a}{n-1}.$
10	a, n, S		$d=\frac{2S-2an}{n(n-1)}.$
11	a, l, S		$d=\frac{l^2-a^2}{2S-l-a}.$
12	n, l, S		$d=\frac{2nl-2S}{n(n-1)}.$
13	d, n, l	a	$a=l-(n-1)d.$
14	d, n, S		$a=\frac{S}{n}-\frac{(n-1)d}{2}.$
15	d, l, S		$a=\frac{1}{2}d\pm\sqrt{{}^{2}(l+\frac{1}{2}d)^2-2dS}.$
16	n, l, S		$a=\frac{2S}{n}-l$
17	a, d, l	n	$n=\frac{l-a}{d}+1.$
18	a, d, S		$n=\pm\frac{\sqrt{(2a-d)^2+8dS}-2a+d}{2d}.$
19	a, l, S		$n=\frac{2S}{l+a}.$
20	d, l, S		$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}.$

SECTION XIX.

GEOMETRICAL PROGRESSION, OR PROGRESSION BY QUOTIENT.

Art. 265. When there are three or more numbers, and the same quotient is obtained by dividing the second by the first, and the third by the second, and the fourth by the third, &c.; or if those quantities or numbers are made to increase by a constant multiplier of the same number, they are said to be in Geometrical Progression, and are called a Geometrical Series. Thus

(1) (2) (3) (4) (5) (6)
(1.) 2, 6, 18, 54, 162, 486 = 728, sum of the series.
(2.) 486, 162, 54, 18, 6, 2 = 728, sum of the series.

The first is called an ascending series, and the second a descending series.

In the first the quotient or multiplier is 3, and it is called the *ratio*. In the second the ratio is $\frac{1}{3}$.

266. The first and last terms of a series are called the *extremes*, and the others are the means.

267. It will readily be perceived in either of the above series, that the product of the extremes is equal to the product of any two of the means equally distant from the extremes. Thus, $2\times486=6\times162=18\times54=972$.

268. If there be only three terms, the product of the extremes is equal to the square of the second term.

269. It is evident by examining either the above series, that any term may be obtained by multiplying the first term by the ratios as many times wanting one as the term required.

If therefore the 1st term was 2, and the ratio 3, and we wished to obtain the 6th term, we have only to multiply the 1st term, 2, by the ratio 3 five times.

Thus, $2\times3\times3\times3\times3\times3=486$, the 6th term.

The above may be generalized in the following manner.

Let a = first term of a series,
L = the last term,
r = the ratio,
n = the number of terms,
S = the sum of the series.

Then $\overset{(1)}{a}$, $\overset{(2)}{ar}$, $\overset{(3)}{ar^2}$, $\overset{(4)}{ar^3}$, $\overset{(5)}{ar^4}$, $\overset{(6)}{ar^5}$, &c. may represent any geometrical series; and, if r the ratio be considered as more than a unit, the series is *ascending;* but, if r is less than a unit, the series is *descending.*

The exponent of r in the second term is 1, in the third term 2, in the fourth term 3, in the fifth term 4, and so on; therefore, the exponent of r in the last term will always be *one* less than the number of terms. The exponent of the nth term in the above series would therefore be ar^{n-1}.

270. If, therefore, in any series, the number of terms be denoted by n, and the last term by L, the following will be the formula for finding the last term.

(1.) $L=ar^{n-1}$,
And $L=r^{n-1}$, when the first term is a unit.

In the above equation we have four quantities, a, L, r, and n; and, if any three of them be given, the others may be obtained as follows.

To find a, the first term, we divide both terms of the above equation by r^{n-1} and transpose the terms, and we have

(2.) $$a=\frac{L}{r^{n-1}}.$$

To obtain r the ratio, we divide the terms of the 1st equation by a, extract the $(n-1)$th root, and transpose the terms, and we have

$$(3.)\qquad r=\sqrt[n-1]{\frac{L}{a}}.$$

To find n, we shall show, when we come to treat of exponential quantities.

EXAMPLES.

1. If the first term be 7, the ratio 3, and the number of terms 5, required the last term.

$$L=ar^{n-1}=7(3)^4=567.\ \textit{Ans.}$$

2. If the first term be 1, the ratio 5, and the number of terms 5, what is the last term?

$$L=r^{n-1}=5^4=625.\ \textit{Ans.}$$

3. If the last term be 405, the ratio 3, and the number of terms 5, what is the 1st term?

$$a=\frac{L}{r^{n-1}}=\frac{405}{3^{5-1}}=5.\ \textit{Ans.}$$

4. If the last term be 8, ratio 5, and the number of terms 4, what is the first term?

$$a=\frac{L}{r^{n-1}}=\frac{8}{5^{4-1}}=\frac{8}{125}.\ \textit{Ans.}$$

5. If the 1st term is 5, the last term 1215, and the number of terms 6, what is the ratio?

$$r=\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{1215}{5}\right)^{\frac{1}{6-1}}=243^{\frac{1}{5}}=3.\ \textit{Ans.}$$

6. If the 1st term is $\frac{1}{5}$, the last term $\frac{27}{320}$, and the number of terms 4, what is the ratio?

$$r=\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{\frac{27}{320}}{\frac{1}{5}}\right)^{\frac{1}{4-1}}=\left(\tfrac{27}{320}\times\tfrac{5}{1}\right)^{\frac{1}{4-1}}=\left(\tfrac{27}{64}\right)^{\frac{1}{3}}=\tfrac{3}{4}.\ \textit{Ans.}$$

7. If the first term be $\frac{1}{16}$, the last term 64, and the number of terms 6, required the ratio. *Ans.* 4.

8. If the last term be 135, the number of terms 4, the ratio 3, what is the first term? *Ans.* 5.

271. To find any number of geometrical means between any two given numbers.

In the 3d formula, we found $r=\sqrt[n-1]{\frac{L}{a}}$.

If we let m represent the number of means, then $m+2=n$; for the number of terms is always two more than the number of means.

Therefore, $\left(\frac{L}{a}\right)^{\frac{1}{n-1}}=\left(\frac{L}{a}\right)^{\frac{1}{m+1}}$.

Consequently, $r=\left(\frac{L}{a}\right)^{\frac{1}{m+1}}$.

272. Having therefore the extremes given to find any number of means, we divide the greater extreme or number by the less extreme, and extract that root of the quotient denoted by the number of means plus 1. This root is the ratio; and having the ratio, the means are readily obtained.

EXAMPLES.

9. Find two geometrical means between 6 and 162.

$162\div6=27$: $\sqrt[3]{27}=3$, the ratio; $6\times3=18$, the first mean; $18\times3=54$, the second mean.

10. What is the geometrical mean between 18 and 882?

$882\div18=49$: $\sqrt{49}=7$, the ratio; $18\times7=126$, the geometrical mean.

11. Required the five geometrical means between 1 and 64. *Ans.* 2, 4, 8, 16, 32.

12. A has a piece of land, which is 18 rods wide, and 288 rods long. Required the side of a square piece, that shall contain an equal quantity of square rods. *Ans.* 72 rods.

273. To find the sum of all the terms of a geometrical series.

Let the following be the series.

(1.) 2, 6, 18, 54, 162.

By examining this series, we find the first term 2, the ratio 3, and the last term 162.

If we multiply each term in the series by the ratio 3, we obtain

(2.) 6, 18, 54, 162, 486.

It is evident, that the sum of this last series is three times the former, therefore the difference between them will be equal to twice the sum of the first series. Thus,

From 6, 18, 54, 162, 486, second series,
Take 2, 6, 18, 54, 162, first series.
—2 486=484, difference of the series.

From the above operation, it appears, that 484 is twice the sum of the first series; and, therefore, $484 \div 2 = 242$ is the sum required.

By examining the process, we perceive that 242 is obtained by multiplying the last term of the first series, 162, by the ratio 3, and subtracting from the product the first term 2, and dividing the remainder, 484, by 2, a number which is *one less* than the ratio. Hence the propriety of the following

RULE. *Multiply the last term by the ratio, find the difference between this product and the first term, divide this remainder by the difference between the ratio and unity, and we have the sum of the series.*

274. We may generalize the above, as follows:

Let a represent the first term of a geometrical series, r the ratio, L the last term, n the number of terms, and S the sum of the series. Then

(1.) $S = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$

We next multiply each term of the above equation by r, and we have,

(2.) $$Sr=ar+ar^2+ar^3+ar^4+ar^5+ar^6.$$

By subtracting the first equation from the second, we have,

$$Sr-S=ar^6-a.$$

Dividing by $r-1$, we have the formula for finding the sum of the series.

$$S=\frac{ar^6-a}{r-1}, \text{ or } \frac{ar^n-a}{r-1}.$$

If the ratio be less than a unit, we transpose the terms, thus,

$$S=\frac{a-ar^6}{1-r}, \text{ or } \frac{a-ar^n}{1-r}.$$

275. The index of the ratio is always equal to the number of terms.

By the above formulæ, we have a method for finding the sum of the series without the last term, which may be expressed by the following

RULE. *Raise the ratio to a power whose exponent is equal to the number of terms, multiply this power by the first term, find the difference between this product and the first term, and divide this remainder by the difference between the ratio and unity.*

If we substitute the value of L as found in Art. 270, we shall have

$$S=\frac{Lr-a}{r-1}.$$

A rule for this formula would be the same as in Art. 274.

13. If the first term be 7, the ratio 3, and the number of terms 5, what is the sum of the series?

$$S=\frac{ar^n-a}{r-1}=\frac{7\times3^5-7}{3-1}=847. \text{ Ans.}$$

14. If the first term be 9, the ratio $\frac{2}{5}$, and the number of terms 4, what is the sum of the series?

$$S=\frac{a-ar^n}{1-r}=\frac{9-(9\times(\frac{2}{5})^4}{1-\frac{2}{5}}=14\tfrac{29}{125}. \text{ Ans.}$$

15. If the first term be 144, the ratio 1.06, and the number of terms 4, what is the sum of the series? *Ans.*

16. If the first term be 9, the ratio $\frac{1}{4}$, the number of terms 6, what is the sum of the series? *Ans.*

17. If the first term be a, the ratio r, and the number of terms n, required the sum of the series.

Ans. $\frac{ar^n-a}{r-1}=\frac{a(r^n-1)}{r-1}$.

18. If the first term be 1, the ratio 2, and the number of terms 7, what is the sum of the series? *Ans.* 127.

19. If the first term is 5, the ratio 10, and the number of terms 7, what is the sum of the series? *Ans.* 55555555.

20. If the first term be 4, the ratio $\frac{1}{4}$, and the number of terms 5, what is the sum of the series? *Ans.* $5\frac{21}{64}$.

21. If the first term be 5, the ratio $\frac{1}{5}$, and the number of terms 5, what is the sum of the series? *Ans.* $6\frac{62}{250}$.

22. A gentleman agreed with another to board him for 10 days. He was to pay 3 cents for the first day's board, 9 cents for the second day, 27 cents for the third day, and so on, in this ratio. What was the amount of the bill for the gentleman's board? *Ans.* $295.24.

To find L, r and a, from the following equation.

$$S=\frac{Lr-a}{r-1}.$$

Multiplying by $r-1$, $Sr-S=Lr-a$,

Resolving into factors, $S(r-1)=Lr-a$,

Transposition, $Lr=S(r-1)+a$,

Division, $L=\frac{S(r-1)+a}{r}$

To find r from the above equation.

$$S=\frac{Lr-a}{r-1}.$$

Multiplying by $r-1$, $$Sr-S=Lr-a,$$

Transposing, $$Sr-Lr=S-a,$$

Dividing by $S-L$, $$r=\frac{S-a}{S-L}.$$

To find a from the above equation.

$$S=\frac{Lr-a}{r-1}.$$

Multiplying by $r-1$, $$Sr-S=Lr-a,$$

Transposing, $$a=Lr-(r-1)S.$$

23. If the first term be 3, the ratio 2, and the sum of the series 93, what is the last term? *Ans.* 64.

24. Insert three geometrical means between $\frac{1}{2}$ and 128. *Ans.* 2, 8, 32.

25. The product of three terms in geometric progression is 64, and the sum of their cubes is 584. What are those numbers? *Ans.* 2, 4, 8.

26. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers. *Ans.* 1, 3, 9, 27.

The following are the two fundamental equations from which the twenty different cases are exhibited.

$$L=ar^{n-1}$$

$$S=\frac{Lr-a}{r-1},$$

and which are found in the following

TABLE.

NO.	GIVEN.	REQUIRED	FORMULÆ.
1	a, r, n	l	$l = ar^{n-1}$.
2	a, r, S		$l = \frac{a+(r-1)S}{r}$.
3	a, n, S		$l(S-l)^{n-1} = a(S-a)^{n-1}$.
4	r, n, S		$l = \frac{(r-1)Sr^{n-1}}{r^n-1}$.
5	a, r, n	S	$S = \frac{ar^n-a}{r-1}$.
6	a, r, l		$S = \frac{lr-a}{r-1}$.
7	a, n, l		$S = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
8	r, n, l		$S = \frac{lr^n-l}{r^n-r^{n-1}}$.
9	a, n, l	r	$r = \sqrt[n-1]{\frac{l}{a}}$.
10	a, n, S		$ar^n - rS = a - S$.
11	a, l, S		$r = \frac{S-a}{S-l}$.
12	n, l, S		$(S-l)r^n - Sr^{n-1} = -l$.
13	r, n, l	a	$a = \frac{l}{r^{n-1}}$.
14	r, n, S		$a = \frac{(r-1)S}{r^n-1}$.
15	r, l, S		$a = lr-(r-1)S$.
16	n, l, S		$a(S-a)^{n-1} = l(S-l)^{n-1}$.
17	a, r, l	n	$n = \frac{\log. l - \log. a}{\log. r} + 1$.
18	a, r, S		$n = \frac{\log. [a+(r-1)S] - \log. a}{\log. r}$.
19	a, l, S		$n = \frac{\log. l - \log. a}{\log. (S-a) - \log. (S-l)} + 1$.
20	r, l, S		$n = \frac{\log. l - \log. [lr-(r-1)S]}{\log. r} + 1$.

The last four cases in the above table can be performed only by the aid of logarithms, as they belong to exponential or transcendental equations. They will therefore receive attention in their proper place.

SECTION XX.

HARMONICAL PROGRESSION.

ART. **276.** Three numbers are said to be in harmonical progression, when the first is to the third, as the difference between the first and second is to the difference between the second and third.

Thus, the numbers 3, 4, 6 are in harmonical proportion.

For, $3 : 6 :: 4-3 : 6-4$.

Or, a, b, c are in harmonical proportion, when

$$a : c :: b-a : c-b.$$

Thus, if the length of three strings of a musical instrument be as the numbers 3, 4, 6, they will sound an octave 3 to 6, a fifth 2 to 3, and a fourth 3 to 4.

277. Four numbers are in harmonical proportion, when the first is to the fourth, as the difference between the first and second is to the difference between the third and fourth. Thus, the numbers 5, 6, 8, 10 are in harmonic proportion.

For, $5 : 10 :: 6-5 : 10-8$.

Strings of such lengths will sound an octave 5 to 10; a sixth greater 6 to 10; a third greater 8 to 10; a third less 5 to 8; and a fourth 6 to 8.

278. Any number of quantities a, b, c, d, e, &c., are in harmonical progression, if $a : c :: a-b : b-c$; $b : d :: b-c : c-d$; $c : e :: c-d : d-e$, &c.

279. The reciprocal quantities in harmonical progression are in arithmetical progression.

Thus, if a, b, c, d, e, &c., are in harmonical progression, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., will be in arithmetical progression.

SECTION XXI.

INFINITE SERIES.

Art. **280.** An infinite decreasing geometrical series is one whose ratio is less than unity, and the number of whose terms is infinite.

To find the sum of an infinite series decreasing in geometrical progression.

We have already found, Art. 274, that the sum of a descending series in geometrical progression by the following formula.

$$S=\frac{a-ar^n}{1-r}\text{; or } S=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

281. Now, if r^n be a fraction less than a unit, it is evident, that the greater the number n, the smaller will be the quantity r^n. If, therefore, a great number of terms of a descending series be taken, the quantity, r^n, will be very small; and if we suppose n greater than any assignable number, then the quantity, or its value, may be considered as nothing $=0$.

Hence the latter part of the formula, $-\frac{ar^n}{1-r}$, should be omitted, and it will stand

Thus, $$S=\frac{a}{1-r}.$$

The rule, therefore, for finding the sum of the series is as follows.

Rule. *Divide the first term by the difference between unity and the ratio.*

EXAMPLES.

1. What is the sum of the infinite series, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c.?

$$\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=1\tfrac{1}{2}.\ \textit{Ans.}$$

2. What is the sum of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. to infinity? *Ans.* 2.

3. What is the sum of the series 8, $\frac{8}{5}$, $\frac{8}{25}$, $\frac{8}{125}$, &c. carried to infinity? *Ans.* 10.

4. Find the value of $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{12}$, &c. to infinity. *Ans.* $1\frac{1}{3}$.

5. Find the value of 4, 1, $\frac{1}{4}$, $\frac{1}{16}$, &c. to infinity. *Ans.* $5\frac{1}{3}$.

6. What is the exact sum of 1, $\frac{1}{10}$, $\frac{1}{100}$, &c. to infinity? *Ans.* $1\frac{1}{9}$.

7. Find the exact value of the circulating decimal .444, &c. to infinity.

$$.444,\ \&c. = \tfrac{4}{10}+\tfrac{4}{100}+\tfrac{4}{1000},\ \text{the ratio being } \tfrac{1}{10}.$$

$$\frac{\frac{4}{10}}{1-\frac{1}{10}}=\frac{\frac{4}{10}}{\frac{9}{10}}=\tfrac{4}{10}\times\tfrac{10}{9}=\tfrac{40}{90}=\tfrac{4}{9}.\ \textit{Ans.}$$

(See National Arithmetic, page 128.)

8. What common fraction will exactly express the value of the repeating decimal .454545, &c.?

$$.454545=\tfrac{45}{100}+\tfrac{45}{10000}+\tfrac{45}{1000000},\ \text{the ratio being } \tfrac{1}{100}.$$

$$\frac{\frac{45}{100}}{1-\frac{1}{100}}=\frac{\frac{45}{100}}{\frac{99}{100}}=\tfrac{45}{100}\times\tfrac{100}{99}=\tfrac{4500}{9900}=\tfrac{5}{11}.\ \textit{Ans.}$$

9. What common fraction is the exact value of the decimal $.\dot{5}7142\dot{8}$? *Ans.* $\frac{4}{7}$.

10. What common fraction is the exact value $.\dot{8}5714\dot{2}$? *Ans.* $\frac{6}{7}$.

11. What is the exact value of $.5\dot{3}$?

$$.5\dot{3}=\tfrac{5}{10} \text{ and } \tfrac{3}{100}+\tfrac{3}{1000}+\tfrac{3}{10000}, \text{ \&c.}$$

$$\frac{\frac{3}{100}}{1-\frac{1}{10}}=\frac{\frac{3}{100}}{\frac{9}{10}}=\tfrac{3}{100}\times\tfrac{10}{9}=\tfrac{30}{900}=\tfrac{1}{30}\,;\ \tfrac{5}{10}+\tfrac{1}{30}=\tfrac{8}{15}. \quad \textit{Ans.}$$

12. What is the value of $.13\dot{8}$? *Ans.* $\frac{5}{36}$.

13. Find the value of an infinite series, whose first term is 8, and the sum of the series 10. *Ans.* $\frac{1}{5}$.

14. Find the ratio of an infinite series, whose first term is $\frac{2}{3}$, and whose sum is $1\frac{1}{3}$. *Ans.* $\frac{1}{2}$.

15. Find the first term of an infinite progression, of which the ratio is $\frac{1}{5}$, and the sum 10. *Ans.* 8.

SECTION XXII.

SIMPLE INTEREST.

Art. 282. Interest is the compensation which the borrower makes to the lender, for the use of a certain sum of money for a given time.

Principal is the sum lent.

Rate per cent. is the sum agreed on for the loan of **\$1**, or **\$100** for one year.

Amount is the sum of the interest and principal.

Legal interest is the rate per cent. established by law.

Let $p=$ principal,

$r=$ rate per cent. written in hundredths,

$t=$ time in years,

$a=$ amount,

i or $a-p=$ interest for the given time.

Hence, if r be the interest of one dollar for one year, it is evident that the interest of p dollars will be p times $r=pr$.

And, if pr be the interest of p dollars for one year, it is certain, that for t years it will be t times as much $=ptr$, and that $p+ptr$ will be the amount, and i or $a-p$ will be the interest.

283. Hence, having the principal, rate per cent. and time given to find the interest and amount, we have the following formulæ.

Formula for the interest,

$$i=ptr.$$

Formula for the amount,

$$a=p+ptr.$$

From the above formulæ we have for finding the interest and amount the following

RULE. *Multiply the principal by the rate per cent. considered as a decimal, and this product by the time in years, and the result is the interest.*

If there be months and days, let the months be considered as fractions of a year, and the days as fractions of a month.

By adding the interest to the principal, we have the amount.

EXAMPLES.

1. What is the interest of $740 for 4 years, at 6 per cent.?

$$i=ptr=740\times.06\times4=\$177.60. \quad \textit{Ans.}$$

2. What is the interest of $380 for 10 years, at 5 per cent.?
Ans. $190.

3. What is the interest of $890.75 for 3 years 6 months, at 8 per cent.? *Ans.* $249.41.

4. What is the interest of $17.18 for 5 years, 2 months, 10 days, at 4½ per cent.? *Ans.* $3.99.

5. What is the amount of $144 for 3 years, at 8 per cent.?

$$a=p+prt=144+(144\times.08\times3)=\$178.56. \quad \textit{Ans.}$$

6. What is the amount of \$800 for 6 years, 1 month, 12 days, at 6 per cent.? *Ans.* \$1093.60.

7. What is the amount of \$670.18 for 3 years, 7 months, 20 days, at 9 per cent? *Ans.* \$889.66.

284. Having the amount, time, and rate per cent. given, to find the principal.

By transposing, &c. the last equation, we have

$$p=\frac{a}{1+tr}.$$

From which we have the following

Rule. *Multiply the time by the rate per cent. and add* 1 *to the product; with this sum divide the amount, and the quotient is the principal.*

8. Received \$472 for a certain sum, that had been on interest at 6 per cent. for 3 years. What was the sum lent?

$$p=\frac{a}{1+tr}=\frac{472}{1+(3\times.06)}=\$400. \quad \textit{Ans.}$$

9. What principal will amount to \$570 in 10 years, at 5 per cent.? *Ans.* \$380.

10. What principal will amount to \$1140.16 in 3 years, 6 months, at 8 per cent.? *Ans.* \$890.75.

11. Lent a certain sum for 5 years, 2 months, 10 days, at $4\frac{1}{2}$ per cent., and received interest and principal \$21.19; what was the sum lent? *Ans.* \$17.18.

12. My friend borrowed of me a certain sum, which he kept 3 years, and for which I charged him 8 per cent., and received interest and principal \$178.56. What was the sum I lent him? *Ans.* \$144.

13. Received as interest and principal \$889.66 from a friend to whom I had loaned a certain sum for 3 years, 7 months, and 20 days, at 9 per cent. What was the consideration of his note? *Ans.* \$670.18.

285. Having the amount, principal, and rate per cent. given, to find the time.

By transposing and reducing the last equation, we have the following formula for finding the time, t.

$$t=\frac{a-p}{rp}=\frac{i}{rp}.$$

From the above formula, we have the following

Rule. *Divide the interest by the product of the principal multiplied by the rate per cent., and the quotient is the time.*

14. How long will it require $300 to amount to $372 at 6 per cent.?

$$t=\frac{a-p}{rp}=\frac{372-300}{.06\times 300}=4 \text{ years. } \textit{Ans.}$$

15. In what time will $380 amount to $570 at 5 per cent.?
Ans. 10 years.

16. Lent at 8 per cent. $890.75, for which I received $1140.16; for how long time was the money lent?
Ans. 3 years, 6 months.

17. For $17.18, which were loaned at 4½ per cent., there were received $21.19. For how long time had it been lent?
Ans. 5 years, 2 months, 10 days.

18. The interest and principal on a certain sum at 9 per cent. are $889.66; and the interest is $670.18 less than the amount. How long was the money at interest?
Ans. 3 years, 7 months, 20 days.

19. A has B's note, dated January 1, 1851, for $320 at 9 per cent. When will the note amount to $353.60?
Ans. March 1, 1852.

286. Having the principal, interest, and time given, to find the rate per cent.

By transposing the last formula, we obtain the following for finding r, the rate per cent. Thus,

$$r=\frac{a-p}{pt}, \text{ or } \frac{i}{pt}.$$

The pupil will perceive, that the amount is known, when the interest and principal are given.

What is the rate per cent. for \$300, that it shall amount to \$372 in 4 years?

$$r=\frac{a-p}{pt}=\frac{372-300}{300\times4}=.06, \text{ or 6 per cent.}$$

Hence we deduce the following

RULE. *Divide the interest by the product of the principal multiplied by the time, and the quotient is the rate per cent.*

20. If \$380 amount to \$570 in ten years, what is the rate per cent.? *Ans.* 5 per cent.

21. Lent \$890.75 for 3 years 6 months, and received for the amount \$1140.16. What was the rate per cent.? *Ans.* 8 per cent.

22. If \$17.18 amount to \$21.19 in 5 years, 2 months and ten days, what is the rate per cent.? *Ans.* 4½ per cent.

23. If the interest of \$670.18 for 3 years, 7 months and 20 days, be \$219.48, what is the rate per cent.? *Ans.* 9 per cent.

24. John Smith, Jr. gave me his note, dated January 1, 1848, for \$144; but he, having been unfortunate in business, I agreed, May 7, 1851, to give him up his note for \$153.64.8. What per cent. did I receive? *Ans.* 2 per cent.

25. My tailor informs me that my "freedom suit" will require 7½ square yards of cloth. But the cloth I am about to purchase will shrink 5 per cent. in width and 4 per cent. in length, and the cloth is 60 inches wide. How many yards must I purchase? *Ans.* 4 yards, 33$\frac{1}{1}\frac{3}{9}$ inches.

SECTION XXIII.

DISCOUNT AT SIMPLE INTEREST.

Art. **287.** Discount is an allowance for the payment of any sum of money before it becomes due, and is the difference between that sum and its present worth.

The present worth of any sum, due some time hence, is such a sum, as, if put at interest, would, in the time for which the discount is to be made, amount to the sum then due.

To find the worth of any sum due at any time hence.

Let $S =$ the sum due,
$p =$ the present worth,
$t =$ the time in years,
$r =$ the rate per cent. considered as so many hundredths.

We have before shown in Art. 283 that $a=p+ptr$.

We now substitute S for a, and consider p to represent the present worth; and, by transposing the equation, find

$$p=\frac{S}{1+tr};$$

from which we deduce the following

Rule. *Multiply the time by the rate per cent., add* 1 *to the product, and divide the sum on which the discount is to be taken by this sum, and the quotient is the present worth.*

If the present worth be taken from the sum due, the remainder is the discount.

1. What is the present value of $500, due 4 years hence, at 6 per cent.?

$$p=\frac{S}{1+tr}=\frac{500}{1+(4\times.06)}=\$403.22+. \quad \textit{Ans.}$$

By transposing the quantities in the above formula, we may obtain the values of s, t, and r.

2. What is the present worth of $372, due 4 years hence at 6 per cent.? *Ans.* $300.

3. What is the present worth of $133.20, due 20 months hence at 8¼ per cent.? *Ans.* $117.09.

4. What is the discount on $21.20, due 5 years, 2 months, 10 days hence, at 4½ per cent.? *Ans.* $4.02.

5. A has B's note dated January 1, 1851, for $353.60, to be paid March 1, 1852, without interest. What was the value of this note at the time it was given, if 9 per cent. discount be allowed? *Ans.* $320.

6. Which is worth the most, A's note for $144, due 10 years hence, at 6 per cent.; or B's note for $176.40, due 8 years hence, at 12 per cent.? *Ans.*

7. A legacy of $1725 is due one year hence. What is its present value, at 15 per cent.? *Ans.* $1500.

8. James Brown has S. Smith's note for $162, payable 6 months hence, but Brown being obliged to raise money, sold the note for $150. What per cent. did he allow? *Ans.* 16 per cent.

9. Bought a farm for $590, for which I was to pay in a certain time, without interest; but, by making prompt payment, I was allowed a discount of 6 per cent. for the whole time, and paid only $500. How long was the time allowed for payment? *Ans.* 3 years.

10. Bought a horse for $200, and gave my note payable in 60 days. What ready money at 15 per cent. will discharge the debt? *Ans.* $195.12+.

11. What is the present worth of $1727 due 100 years hence, at 6 per cent.? *Ans.* $261.

SECTION XXIV.

PARTNERSHIP, OR COMPANY BUSINESS.

ART. 288. Partnership is the association of two or more persons in business, with an agreement to share the profits and losses in proportion to the amount of the capital stock contributed by each.

EXAMPLES.

1. Three men, A, B, and C, enter into partnership for two years with a capital of $1600. A puts into the firm $300, B $500, and C $800. They gain $320. What is each man's share of the gain?

Let n=A's gain.

Then, as each man's share of the gain will be in proportion to his stock,

And $\frac{5x}{3}$=B's gain.

$\frac{8x}{3}$=C's gain.

And
$$x+\frac{5x}{3}+\frac{8x}{3}=\$320,$$
$$3x+5x+8x=960,$$
$$16x=960,$$
$$x=60=\text{A's gain},$$
$$\frac{5x}{3}=100=\text{B's gain},$$
$$\frac{8x}{3}=160=\text{C's gain}.$$

VERIFICATION.

$$60+100+160=\$320.$$

Or, let m, n, and p represent A, B, and C's stock; and a the sum gained.

Also, let x=A's gain.

Then, it is evident, that each man must receive according to his capital.

That is, as A's stock is to his gain, so will B's stock be to his gain, &c.

Therefore, $m:x::n:\frac{nx}{m}$=B's,

And $m:x::p:\frac{px}{m}$=C's.

Then, $x+\frac{nx}{m}+\frac{px}{m}=a$,

And $mx+nx+px=am$;

Therefore, $x=\frac{am}{m+n+p}=\frac{320\times300}{300+500+800}$=\$60, A's gain.

Then, by the principle above stated,

$m:\frac{am}{m+n+p}::n:\frac{an}{m+n+p}=\frac{320\times500}{300+500+800}$=\$100, B's gain.

And,

$m:\frac{am}{m+n+p}::p:\frac{ap}{m+n+p}=\frac{320\times800}{300+500+800}$=\$160, C's gain.

VERIFICATION.

$$\frac{am}{m+n+p}+\frac{an}{m+n+p}+\frac{ap}{m+n+p}=\frac{(m+n+p)a}{m+n+p}=a=\$320.$$

Therefore, to find the gain or loss on any man's stock, we deduce from the above formulæ the following

RULE. *Multiply the whole gain by each man's stock, and divide the product by the whole stock.*

289. Having each man's gain and the amount of stock given, to find each man's share in the stock.

2. A, B and C, while in trade, gained as follows. A gained \$50, B \$70, and C \$90. The amount of their stock in trade was \$4200. What was the amount of each man's stock?

It is evident that each man's stock was in proportion to his gain.

Let $x =$ A's stock,

Then $\frac{7x}{5} =$ B's stock,

And $\frac{9x}{5} =$ C's stock.

$$\text{Therefore,}\quad x+\frac{7x}{5}+\frac{9x}{5}=4200.$$

$$5x+7x+9x = 21000.$$

$$21x = 21000.$$

$$x = 1000.\ \text{A's stock.}$$

$$\frac{7x}{5} = 1400.\ \text{B's stock.}$$

$$\frac{9x}{5} = 1800.\ \text{C's stock.}$$

$$\overline{4200.}\ \text{Proof.}$$

If we change the symbols of the first question, putting m, n, and p, for the gain of each man respectively, and a for the stock, we obtain the following formulæ for finding the amount of each man's stock.

$$\frac{ma}{m+n+p}=\frac{50\times4200}{50+70+90}=\$1000.\ \text{A's stock.}$$

$$\frac{na}{m+n+p}=\frac{70\times4200}{50+70+90}=\$1400.\ \text{B's stock.}$$

$$\frac{pa}{m+n+p}=\frac{90\times4200}{50+70+90}=\$1800.\ \text{C's stock.}$$

Hence for finding each man's stock, we have the following

RULE. *Multiply the whole stock by each man's gain, and divide the product by the whole gain.*

3. Two men, M and N, engaged in trade. M put in $500, and N $750. They gained $120. What is each man's gain?

Ans. M gained $48, N gained $72.

4. Q and X hired a field for $120, which they used for a pasture. Q put in 11 cows, and X 15 cows. What sum should each man pay?

Ans. Q pays $$50.76\frac{12}{13}$, X pays $$69.23\frac{1}{13}$.

5. A and B purchased a factory for $17,000. A paid $10,000, and B the remainder. They gained $1500. What sum should each receive? *Ans.* A $$882\frac{6}{17}$, B $$617\frac{11}{17}$.

6. A, B, and C engaged in trade with a capital of $6000. They gained $240. A's share of the gain was $100, B's $80, and C's $60. What part of the stock did each own?

Ans. A $2500, B $2000, and C $1500.

7. A, B, and C hire a pasture for the season for $100. A put in 5 horses, B 7 oxen, and C 9 cows. Two horses eat as much as 3 oxen, and 4 oxen eat as much as 5 cows. What part of the expense must each pay?

Ans. A pays $$34.56\frac{48}{217}$, B pays $$32.25\frac{175}{217}$, and C pays $$33.17\frac{211}{217}$.

8. Three men, A, B, and C, agreed to reap a field that was 40 rods square for $32. A reaped a part that was 25 rods square, B reaped 400 square rods, and C the remainder. What sum did each receive?

Ans. A $12.50, B $8.00, C $11.50.

PARTNERSHIP ON TIME, OR DOUBLE FELLOWSHIP.

290. 9. A, B, and C engaged in trade. A put in $2000 for 4 months, B put in $3000 for 2 months, and C put in $4000 for 12 months. They gained $780. What is each man's share of the gain?

Let m, m, p represent each man's stock, a the whole gain, and t, t', t'' the time each man's stock was in trade. It is evident that each man's stock gains not only in proportion to its sum, but also in proportion to the time it is in trade. For $2000 will gain four times as much in four months as it would in one month; and $2000 for four months is the same as $8000 for one month. We must therefore multiply each man's stock by the time it was in trade. It is therefore evident, that as A's gain is to B's gain, as A's stock multiplied by his time is to B's stock multiplied by his time, &c.

Let x, y, z = A, B, C's gain respectively.

Then, $$x : y :: mt : nt'.$$

Multiplying extremes, &c. $$y = \frac{nt'x}{mt} = \text{B's gain.}$$

And $$x : z :: mt : pt''.$$

Multiplying extremes, &c. $$z = \frac{pt''x}{mt} = \text{C's gain.}$$

And $$x + \frac{nt'x}{mt} + \frac{pt''x}{mt} = a.$$

Multiplying by mt, $$mtx + nt'x + pt''x = mta.$$

Therefore,

$$x = \frac{mta}{mt + nt' + pt''} = \frac{2000 \times 4 \times 780}{2000 \times 4 + 3000 \times 8 + 4000 \times 12} = \$78.\ \text{A's gain.}$$

But $$y = \frac{nt'x}{mt}.$$

And by substitution, $$y = \frac{nt'}{mt} \times \frac{mta}{mt + nt' + pt''} = \frac{nt'a}{mt + nt' + pt''} =$$

$$\frac{3000 \times 8 \times 780}{2000 \times 4 + 3000 \times 8 + 4000 \times 12} = \$234.\ \text{B's gain.}$$

And $$z = \frac{pt''x}{mt}.$$

And by substitution,

$$z=\frac{pt''}{mt}\times\frac{mta}{mt+nt'+pt''}=\frac{pt''a}{mt+nt+pt''}=$$

$$\frac{4000\times 12\times 780}{2000\times 4+3000\times 8+4000\times 12}=\$468.\text{ C's gain.}$$

The above equations, by dividing the numerators, each, into two factors, may be expressed by the following proportions.

$$mt+nt'+pt'' : mt :: a : x.$$

$$mt+nt'+pt'' : nt' :: a : y.$$

$$mt+nt'+pt'' : pt'' :: a : z.$$

Hence the following arithmetical

RULE. *Multiply each man's stock by the time it was continued in trade, and then say,* As the sum of all the products is to each man's product, so is the whole gain or loss to each man's gain or loss.

(See NATIONAL ARITHMETIC, Sec. LVI.)

10. A commenced business January 1, 1850, with a capital of $3000. May 1, 1850, he took B into partnership, with a capital of $4000. January 1, 1851, they had gained $340. What was each man's share of the gain?

Ans. A's gain $180, B's gain $160.

11. A, B, and C traded in company. A put in $300 for 10 months, B put in $400 for 8 months, and C put in $600 for 2 months. They gained $120. What is the gain of each?

Ans. A's gain 48.64\frac{32}{37}$, B's 51.89\frac{7}{37}$, C's 19.45\frac{35}{37}$.

12. Three men, A, B, and C, hire a pasture in common, for which they are to pay $76.80. A put in 24 oxen for 12 weeks, B put in 25 oxen for 12 weeks, and C put in 30 oxen for 6 weeks. What sum ought each to pay?

Ans. A $28.80, B $30, C $18.

13. John Jones hired a house for one year for $500, with the privilege of admitting two more families if he pleased, with the understanding that each occupant should have equal privileges in the house. At the end of three months he took in John Smith, and at the end of 9 months Richard Roe. What share of the rent should each pay?

Ans. Jones $$291\frac{2}{3}$, Smith $$166\frac{2}{3}$, Roe $$41\frac{2}{3}$.

14. Two men, A and B, hired a coach in Boston to go to Worcester, the distance being 42 miles, for $20, with the privilege of taking in two persons more. Having rode 30 miles, they take in C, and on their return from Worcester, when within 20 miles of Boston, they take in D. What ought each man to pay for his accommodation in the coach?

Ans. A $$7.46\frac{8}{252}$, B $$7.46\frac{8}{252}$, C $$3.88\frac{224}{252}$, D $$1.19\frac{12}{252}$.

15. A and B engage in trade. A puts in *a* dollars for *b* months, B puts in *c* dollars for *d* months, and they gain *e* dollars. What share of the gain shall each receive?

Ans. $\frac{abe}{ab+cd}$ A's gain, $\frac{cde}{ab+cd}$ B's gain.

16. A, B, and C engage in trade with a capital of $1911. A's money was in the firm 3 months, B's 5 months, and C's 7 months. They gained $117, which was so divided, as that the $\frac{1}{2}$ of A's gain was equal to $\frac{1}{3}$ of B's, and $\frac{1}{4}$ of C's gain. What was each man's stock and gain?

Ans. { A's stock $234, B's stock $585, and C's stock $1092.
A's gain $26, B's gain $39, and C's gain $52.

17. If 12 oxen eat $3\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen eat 10 acres in 9 weeks, how long would it require 36 oxen to eat 24 acres, the grass to be growing uniformly?

Ans. 18 weeks.

SECTION XXV.

LOGARITHMS.*

Art. 291. Logarithms are a series of numbers in arithmetical progression, answering to another series of numbers in geometrical progression.

Thus, { 0, 1, 2, 3, 4, 5, 6, indices or logarithms.
1, 2, 4, 8, 16, 32, 63, geometrical progression.

Or, { 0, 1, 2, 3, 3, 5, 6, indices or logarithms.
1, 3, 9, 27, 81, 243, 729, geometrical progression.

Or, { 0, 1, 2, 3, 4, 5, indices or log.
1, 10, 100, 1000, 10000, 100000, geomet. prog.

From the above, it is evident, that the same indices may serve equally for any geometric series; and, consequently, there may be an endless variety of systems of logarithms to the same common numbers, by only changing the second term, 2, 3, or 10, &c. of the geometrical series of whole numbers, and by interpolation the whole system of numbers may be made to enter the geometric series, and receive their proportional logarithms, whether integers or decimals.

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* The invention of Logarithms is due to Lord Napier, Baron of Merchiston, in Scotland, and is properly considered as one of the most useful inventions of modern times. A table of these numbers was first published by the inventor at Edinburgh in the year 1614, in a treatise entitled, *Canon Mirificum Logarithmorum,* which was eagerly read by all the learned throughout Europe. Mr. Henry Briggs, then professor of geometry at Gresham College, soon after the discovery, went to visit the noble inventor; after which, they jointly undertook the arduous task of computing new tables on this subject, and reducing them to a more convenient form than that which was at first thought of. But Lord Napier, dying soon after, the whole burden fell upon Mr. Briggs, who, with prodigious labor and great skill, made an entire canon according to the new form, for all numbers from 1 to 20000, and from 90000 to 101000 to 14 places of decimals, and published it in London in the year 1624.

It is also apparent from the nature of these series, that, if any two indices be added together, their sum will be the index of that number, which is equal to the product of the two terms in the geometric progression to which those indices belong. Thus the indices of 2 and 3 being taken together make 5, and the numbers 4 and 8, or the terms corresponding to those indices, being multiplied together, make 32, which is the number answering to the index 5.

In like manner, if any one index be subtracted from another, the difference will be the index of that number, which is equal to the quotient of the two terms to which those indices belong. Thus the index of 6, minus the index 4, is $=2$; and the terms corresponding to those indices are 64 and 16, whose quotient is $=4$, which is the number answering to the index 2.

For the same reason, if the logarithm of any number be multiplied by the index of its power, the product will be equal to the logarithm of that power. Thus, the index or logarithm of 4, in the above series is 2; and, if this number be multiplied by 3, the product will be $=6$, which is the logarithm of 64, or the third power of 4.

And, if the logarithm of any number be divided by the index of its root, the quotient will be equal to the logarithm of that root. Thus the index or logarithm of 64 is 6; and, if this number be divided by 2, the quotient will be $=3$, which is the logarithm of 8, or the square root of 64.

The logarithms most convenient for practice are such, as are adapted to a geometric series increasing in a tenfold proportion, as in the last of the above forms; and are those, which are to be found, at present, in most of the common tables on this subject. The distinguishing mark of this system of logarithms is, that the index or logarithm of 10 is 1; that of 100, 2; that of 1000, 3, &c.

In decimals the logarithm of .1 is —1; and that of .01 is —2; that of .001 is —3, and so on. The logarithm of 1 in every system being 0, it follows, that the logarithm of any number between 1 and 10 must be 0 and some fractional parts; and that of a number between 10 and 100 will be 1 and some fractional parts; and so on for any other number whatever. And since the integral part of a logarithm, usually called the Index or Characteristic, is always thus readily found, it is commonly omitted in the tables; being left to be supplied by the operator himself, as occasion requires.

292. Another definition of Logarithms is, that the logarithm is the index of that power of some other number, which is equal to the given number. So, if there be $N=r^n$, then n is the logarithm of N; where n may be either positive or negative, or nothing, and the root, r, any number whatever, according to the different systems of logarithms.

When n is $=0$, then N is $=1$, whatever the value of r is, which shows that the logarithm of 1 is always 0 in every system of logarithms. When n is $=1$, then N is $=r$; so that the radix, r, is always that number, whose logarithm is 1, in every system. When the radix $r=2.718281828459$, &c., the indices n are the hyperbolic or Napier's logarithm of numbers N; so that n is always the hyperbolic logarithm of the number N, or $(2.718281828459)^n$.

293. When the radix $r=10$, then the index n becomes the common or Briggs's logarithm of the number N; so that the common logarithm of any number 10^n or N is n, the index of that power of 10, which is equal to the said number. Thus, 100 being the second power of 10, will have 2 for its logarithm; and 1000 being the third power of 10, will have 3 for its logarithm. Hence, also, if 50 be $=10^{1.69897}$, then is 1.69897 the common logarithm of 50. That is, 10 has been raised to the 169897th power, and the 100000d root has been extracted, which is found to be 50 nearly. And, in general, the following decuple series of terms, viz.

10^4, 10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4},

or 100000, 1000, 100, 10, 1, .1, .01, .001, .0001,

have 4, 3, 2, 1, 0, —1, —2, —3, —4,

for their logarithms, respectively. And from this scale of numbers and logarithms, the same properties easily follows, as above mentioned.

294. To compute the Logarithm to any of the Natural Numbers, 1, 2, 3, 4, 5, &c., we have the following

RULE. *Take the geometrical series,* 1, 10, 100, 1000, 10000, *&c., and apply it to the arithmetical series,* 0, 1, 2, 3, 4, 5, *&c. as logarithms.*

Find a geometrical mean between 1 *and* 10, *or between* 10 *and* 100, *or any other two adjacent terms of the series, between which the number proposed lies.*

In like manner between the mean thus found, and the nearest extreme, find another geometrical mean; and so on, till you arrive within the proposed limit of the number, whose number is sought.

Find also as many arithmetical means in the same, as you found geometrical ones, and these will be the logarithms answering to the said geometrical means.

EXAMPLE.

Calculate the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. 10 is 1, and the log. of 1 is 0.

Therefore $(1+0)\div2=\frac{1}{2}=.5$ is the arithmetical mean;

And $(10\times1)^{\frac{1}{2}}=3.1622777$ the geometrical mean;

Hence the log. of 3.1622777 is 5.

Secondly, the log. of 10 is 1, and the log. of 3.1622777 is .5 ;
Therefore $(1+.5)\div2=.75$ the arithmetical mean ;
And $(10\times3.1622777)^{\frac{1}{2}}=5.6234132$ the geometrical mean ;
Hence the log. of 5.6234132 is .75.

Thirdly, the log. of 10 is 1, and the log. of 5.6234132 is .75 ;
Therefore $(1+.75)\div2=.875$ is the arithmetical mean ;
And $(10\times5.6234132)^{\frac{1}{2}}=7.4989422$ the geometrical mean ;
Hence the log. of 7.4989422 is .875.

Fourthly, the log. of 10 is 1, and the log of 7.4989422 is 875 ;
Therefore, $(1+.875)\div2=.9375$ is the arithmetical mean ;
And $(10\times7.4989422)^{\frac{1}{2}}=8.6596431$ the geometrical mean ;
Hence the log. of 8.6596431 is .9375.

Fifthly, the log. of 10 is 1, and the log. of 8.6596431 is 9375.
Therefore, $(1+.9375)\div2=.96875$ is the arithmetical mean ;
And $(10\times8.6596431)^{\frac{1}{2}}=9.3057204$ the geometrical mean ;
Hence the log. of 9.3057204 is .96875.

Sixthly, the log. of 8.6596431 is .9375, and the log. of 9.3057204 is .96875.

Therefore, $(.9375+.96875)\div2=.953125$ is the arithmetical mean ;

And $(8.6596431\times9.3057204)^{\frac{1}{2}}=8.9768713$ the geometrical mean ;

Hence the log. of 8.9768713 is .953125.

By proceeding in this manner, after 25 extractions, it will be found, that the logarithm of 8.9999998 is .9542425, which may be taken for the logarithm of 9, as it differs so little from it, and is sufficiently exact for all practical purposes; and, in this manner, were the logarithms of almost all the prime numbers at first computed.

295. Another method of computing logarithms is by the aid of a given decimal.

Rule. *Let* b *be the number, whose logarithm is required to be found; and* a *the number next less than* b, *so that* $b-a=1$, *the logarithm of* a *being known; and let* s *denote the sum of the two numbers* $a+b$. *Then*

1. *Divide the constant decimal,* .8685889638 *by* s, *and reserve the quotient; divide the reserved quotient by the square of* s, *and reserve this quotient: divide this last quotient, also, by the square of* s, *and again reserve the quotient; and thus proceed continually dividing the last quotient by the square of* s, *as long as division can be made.*

2. *Write these quotients orderly under one another, the first uppermost, and divide them respectively by the odd numbers,* 1, 3, 5, 7, 9, *&c., as long as division can be made; that is, divide the reserved quotient by* 1, *the second by* 3, *the third by* 5, *the fourth by* 7, *and so on.*

3. *Add all these last quotients together, and the sum will be the logarithm of* $b \div a$. *To this logarithm add also the given logarithm of the said next less number,* a; *so will the last sum be the logarithm of the number* b *proposed.*

EXAMPLES.

1. Let it be required to find the logarithm of the number 2.

Here the given number b is 2, and the next less number a is 1, whose logarithm is 0; also, the sum $2+1=3=s$, and its square $s^2=9$. Then, the operation will be as follows.

3)	868588964	1)	289529654	(289529654
9)	289529654	3)	32169962	(10723321
9)	32169962	5)	3574440	(714888
9)	3574440	7)	397160	(56737
9)	397160	9)	44129	(4903
9)	44129	11)	4903	(446
9)	4903	13)	545	(42
9)	545	15)	61	(4
9)	61			

Logarithm of $\frac{2}{1}$=.301029995

Add logarithm of 1=.000000000

Logarithm of 2=.301029995

2. Compute the logarithm of the number 3.

Here $b=3$, the next less number $a=2$, and the sum $a+b=5=s$, whose square $s^2=25$.

5)	.868588964	1)	.173717793	(	.173717793
25)	.173717793	3)	6948712	(	2316237
25)	6948712	5)	277948	(	55590
25)	277948	7)	11118	(	1588
25)	11118	9)	4445	(	50
25)	445	11)	18	(	2
	18				

Logarithm of $\frac{3}{2}$=.176091260
Logarithm of 2 add.=.301029995

Logarithm of 3=.477121255

296. Because the sum of the logarithms of numbers gives the logarithm of their product; and the difference of the logarithms gives the logarithm of the quotient of the number, we may, therefore, from the above two logarithms, and the logarithm of 10, which is 1, raise a great many logarithms, as will appear by the following

EXAMPLES.

1. To find the logarithm of 4, we multiply the logarithm of 2=.301030 by 2, because twice 2 are 4.

Logarithm of 2=.301030
2

Logarithm of 4=.602060

Find the logarithm of 6.

Because 2×3=6, we add their logarithms.

Logarithm of 2=.301030
Logarithm of 3=.477121

Logarithm of 6=.778151

Find the logarithm of 8.

Because $2^3=8$; therefore

Logarithm of 2=.301030
Multiplied by 3= 3
Gives logarithm of 8=.903090

Find the logarithm of 9.

Because $3^2=9$; therefore

Logarithm of 3=.477121
Multiplied by 2= 2
Gives logarithm of 9=.954242

Find the logarithm of 5.

Because $\frac{10}{2}=5$; therefore

From logarithm of 10=1.000000
Subtract logarithm of 2= .301030
Logarithm of 5. *Ans.* .698970

Having computed by the general rule the logarithms to the other prime numbers, 7, 11, 13, 17, 19, 23, &c., then, by composition and division, we may easily find as many logarithms as we please.

NOTE.—The index to every logarithm is always *one less* than the integers of the given number.

297. To find in the table the logarithm of any number.

(1.) If the given number be less than 100, or consists of only two figures.

RULE. *Enter the first page of the table, which contains all the numbers from* 1 *to* 100, *and opposite the given number will be found the logarithm with the index prefixed.*

(2.) If the given number be more than 100, and less than 1000.

Rule. *Find the given number in the left hand column of the table, and opposite, in the next column, will be found the logarithm to which the index*, 2, *must be prefixed.*

Thus, if the logarithm of 189 were required, we find this number in the table, and, opposite to it, we find the logarithm .276462. To this we prefix the index, 2, and we have 2.276462.

(3.) If the given number be more than 1000, and less than 10.000.

Rule. *Find the first three figures of the given number in the left hand column, and, opposite to it, in the column marked at the top with the fourth figure, is the logarithm required. To which must be prefixed the index*, 3.

Thus, if the logarithm of 3568 were required, we find opposite 356 in the left hand column, and under 8 found at the top of the column, .552425. To this we prefix the index, 3, because there are four figures in the given number, thus 3.552425.

(4.) If the given number be more than 10000.

Rule. *Find the logarithm of the first four figures as before; also the next greater logarithm; subtract the one logarithm from the other, as also their corresponding numbers, the one from the other. Then say, As the difference between the two numbers is to the difference of their logarithms, so is the remaining part of the given number to the proportional part of the logarithm; which part, being added to the less logarithm before taken out, gives the whole logarithm nearly.*

EXAMPLES.

1. Find the logarithm of 340926.

The logarithm of	340900	is	= .532627
The logarithm of	341000	is	= .532754
The differences are =	100		127

Then, as 100 : 127 : : 26 : 33, the proportional part. This, added to the first logarithm, (.532627+33) gives .532660. To this we prefix the index 5; because the given number had six figures.

(5.) To find the logarithm of a number consisting of an integer and decimal.

RULE. *Find the logarithm of the decimal part the same as if all its figures were integral; then this, having prefixed to it the proper index, will give the logarithm required, remembering that the index will always be* one less *than the integers.*

Thus, the logarithm of 42.25 is 1.625827.

(6.) To find the logarithm of a proper fraction.

RULE. *Subtract the logarithm of the denominator from the logarithm of the numerator, and the remainder will be the logarithm sought; which being that of a decimal fraction, must always have a negative index.*

2. What is the logarithm of $\frac{37}{94}$?

Logarithm of 37 =1.568202

Logarithm of 94 =1.973128

Logarithm of $\frac{37}{94}$ =—1.595074

(7.) To find the logarithm of a mixed number.

RULE. *Reduce the mixed number to an improper fraction, and find the difference of the logarithms of the numerator and denominator in the same manner as above.*

3. What is the logarithm of $17\frac{14}{23}$?

First $17\frac{14}{23} = \frac{405}{23}$. Then,

Logarithm of 405 = 2.607455

Logarithm of 23 = 1.361728

Logarithm of $17\frac{14}{23}$ = 1.245727

(8.) To find the logarithm of any decimal.

Rule. *Find the logarithm of the decimal as of an integer; and, if the first figure in the decimal occupy the place of tenths, the index will be* —1. *Thus, the logarithm of* .375 *will be* —1.574031. *If the first decimal place occupy the place of hundredths, the index will be* —2. *If the decimal is preceded by two ciphers, the index will be* —3 *; and so on.*

Thus, the logarithm of .234 = —1.369216
of .0234 = —2.369216
of .00234 —3.369216
of .000234 —4 369216
of .0000234 —5.369216

EXAMPLES.

1. What is the logarithm of 1728 ?
2. What is the logarithm of 23.56 ?
3. What is the logarithm of 89632 ?
4. What is the logarithm of $\frac{17}{93}$?
5. What is the logarithm of $\frac{3}{107}$?
6. What is the logarithm of $19\frac{2}{37}$?
7. What is the logarithm of .3076 ?
8. What is the logarithm of .00016 ?
9. What is the logarithm of .0000006 ?

298. To find the natural number to any given logarithm.

This is to be found in the tables by the reverse method to the former, by searching for the proposed logarithm among those in the table, and taking out the corresponding number by inspection, in which the proper number of integers are to be pointed off ; that is, one more than the index. For, in finding the number answering to any given logarithm, the index always shows how far the first figure must be removed from the place of units to the left hand or integers, when the index is affirmative ; but the right hand, or decimals, when it is negative.

Thus the number to the logarithm 1.532882 is 34.11.

And the number of the logarithm —1.532882 is .3411.

But, if the logarithm cannot be exactly found in the table, we adopt the following

RULE. *Take out the next greater and the next less, subtracting one of these logarithms from the other, as also their natural numbers, the one from the other, and the less logarithm from the logarithm proposed. Then say, As the difference of the first, or tabular logarithms, is to the difference of their natural numbers, so is the difference of the given logarithms and the least tabular logarithm to the corresponding numeral difference.*

Which being annexed to the least natural number above taken, gives the natural number sought, corresponding to the proposed logarithm.

EXAMPLE.

1. What is the natural number answering to the given logarithm 1.532708 ?

Next greater 532754, its number 341000; given log. 532708
Next less 532627, its number 340900; next less 532627
127 100 81

Then, as 127 : 100 : : 81 : 64 nearly the numeral difference. Therefore, 340900 + 64 = 34.0964, marking off two integers, because the index of the given logarithm is 1.

Had the index been —1.532708, its corresponding number would have been .340964, wholly a decimal.

MULTIPLICATION OF LOGARITHMS.

RULE. *Take out the logarithms of the factors from the table, then add them together, and their sum will be the loga-*

rithm of the product required. Then take out from the table the natural number, answering to the sum, for the product sought.

Add what is to be carried from the decimal part of the logarithm to the affirmative index or indices, or else subtract it from the negative.

Also, adding the indices together, when they are of the same kind, both affirmative or both negative; but subtracting the less from the greater, when the one is affirmative and the other negatire, and prefixing the sign of the greater to the remainder.

EXAMPLES.

1. Multiply 23.14 by 5.062.

	Numbers.		Logarithms.
	23.14	=	1.364363
	5.062	=	0.704322
Product,	117.1347	=	2.068685

2. Multiply 2.581926 by 3.457291.

	Numbers.		Logarithms.
	2.581926	=	0.411944
	3.457291	=	0.538736
Product,	8.92648		0.950680

3. What is the continued product of 3.902, 597.16, and .0314728 ?

	Numbers.		Logarithms.
	3.902	=	0.591287
	597.16	=	2.776091
	.0314728	=	—2.497935
Product,	73.333	=	1.865313

Here the —2 cancels the +2, and the 1 to carry from the decimal is set down.

What is the continued product of 3.586, 2.1046, 0.8372, and 0.0294.

	Numbers.	Logarithms.
	3.586	0.554610
	2.1046	0.323170
	0.8372	—1.922829
	0.0294	—2.468347
Product,	0.1057618	—1.268956

Here the 2 to carry cancels the —2, and there remains —1 to set down.

DIVISION BY LOGARITHMS.

Rule. *From the logarithm of the dividend subtract the logarithm of the divisor, and the number answering to the remainder will be the quotient required.*

Change the sign of the index of the divisor, from affirmative to negative, or from negative to affirmative, then take the sum of the indices, if they be of the same name, or their difference, when of different signs, with the sign of the greater for the index to the logarithm of the quotient.

And also when 1 is borrowed in the left hand place of the decimal part of the logarithm, add it to the index of the divisor, when that index is affirmative, but subtract it when negative; then let the sign of the index arising from hence be changed, and worked with as before.

EXAMPLES.

1. Divide 24163 by 4567.

Logarithm of 24163 = 4.383151
Logarithm of 4567 = 3.659631

Quotient, 5.29078 = 0.723520

2. Divide 37.149 by 523.76.

Logarithm of 37.149 = 1.569947
Logarithm of 523.76 = 2.719132
Quotient, .0709275 = —2.850815

3. Divide .06314 by .007241.

Logarithm of .06314 = —2.800305
Logarithm of .007241 = —3.859799
Quotient, 8.71979 = 0.940506

Here 1 carried from the decimals to the —3, makes it become —2, which taken from the other —2, leaves 0 remainder.

4. Divide .7438 by 12.9476.

Logarithm of .7438 = —1.871456
Logarithm of 12.9476 = 1.112189
Quotient, .057447 —2.759267

Here 1 taken from the —1, makes it become —2 to set down.

299. To find the Arithmetical Complement of the logarithm of any number.

RULE. *Subtract the logarithm of the number from the logarithm of* 1, *which is zero* (0).

EXAMPLES.

1. What is the arithmetical complement of 1.462398?

0.
1.462398
—2.537602

2. What is the arithmetical complement of —1.397940?

0.
—1.397940
0.602060

3. What is the arithmetical complement of —3.678914?

$$\begin{array}{r} 0. \\ -3.678914 \\ \hline 2.321086 \end{array}$$

4. What is the arithmetical complement of 3.614582?

$$\begin{array}{r} 0. \\ 3.614582 \\ \hline -4.385418 \end{array}$$

5. What is the arithmetical complement of —4.321617?

6. What is the arithmetical complement of 0.781562?

7. What is the arithmetical complement of 5.321463?

8. What is the arithmetical complement of 3.456321?

The pupil will understand the rationale of this rule by observing, that the product of a multiplied by b is the same as a divided by $\frac{1}{b}$.

Thus, $a \times b = ab$, or $a \div \frac{1}{b} = ab$.

Or, 12 multiplied by 5 is the same as 12 divided by $\frac{1}{5}$.

Thus, $12 \times 5 = 60$; or $12 \div \frac{1}{5} = 60$.

The same by logarithms.

Logarithm of 12	=1.079181
Logarithm of 5	=0.698970
Logarithm of the product, 60	=1.778151.

Or,

Logarithm of 12	=1.079181
Logarithm of $\frac{1}{5}$=.2=—1.301030 Arith. Com.	=0.698970
Logarithm of the product, 60	=1.778151.

300. Any number may be divided by adding the arithmetical complement of the divisor to the logarithm of the dividend. Their sum will give the logarithm of the quotient.

9. Divide 1728 by 12.

Logarithm of 1728 =3.237544
Logarithm of 12=1.079181 Arith. Com. =—2.920819

Ans. 144=2.158363

10. What is the value of x in the following equation?

$$x=\frac{1728\times144\times6}{36\times18\times12}.$$

Log. 1728 = 3.237544
Log. 144 = 2.158362
Log. 6 = 0.778151
Log. 36=1.556303 Arith. Com. =—2.443697
Log. 18=1.255273 " " =—2.744727
Log. 12=1.079181 " " =—2.920819

Ans. 192=2.283300

11. What is the value of x in the following equation?

$$x=\frac{48\times.75\times72\times.0625}{.027\times120}.$$

Log. 48 = 1.681241
Log. .75 =—1.875061
Log. 72 = 1.857332
Log. .0625 =—2.795880
Log. .027=—2.431364 Arith. Com. = 1.568636
Log 120= 2.079181 " " =—3.920819

Ans. 50=1.698969

12. What is the value of x in the following equation?

$$x=\frac{654\times320\times.3691}{87\times9\times.045}.$$

Ans.

13. What is the value of x in the following equation?

$$x=\frac{.69\times7.5\times32.71\times.003}{87\times8968\times.0008}.$$

Ans.

14. Multiply eight thousand and thirty-six thousandths, by eighty-nine and nine thousandths, and divide this product by three hundred nine hundred thousandths. *Ans.*

INVOLUTION BY LOGARITHMS.

RULE. *Take out the logarithm of the given number from the table. Multiply the logarithm thus found by the index of the power proposed. Find the number answering to the product, and it will be the power required.*

NOTE. — In multiplying a logarithm with a negative index by an affirmative number, the product will be negative. But that, which is to be carried from the decimal part of the logarithm, will be affirmative. And therefore these differences will be the index of the product, and is always to be made of the same kind with the greater.

EXAMPLES.

1. What is the square of 2.579?

$$\begin{array}{lrcr} \text{Logarithm of} & 2.579 & = & 0.411439 \\ & & & 2 \\ \hline & \textit{Ans. } 6.646+ & = & 0.822878 \end{array}$$

2. What is the third power of 32.16?

$$\begin{array}{lrcr} \text{Logarithm of} & 32.16 & = & 1.507316 \\ & & & 3 \\ \hline & \textit{Ans. } 33261.9 & = & 4.521948 \end{array}$$

3. Required the fourth power of 0.9163.

$$\begin{array}{lrcr} \text{Logarithm of} & .09163 & = & -2.962038 \\ & & & 4 \\ \hline & \textit{Ans. } .000070494 & = & -5.848152 \end{array}$$

Here 4 times the negative index being —8, and 3 to carry, the difference —5 is the index of the power.

EVOLUTION BY LOGARITHMS.

Rule. *Take the logarithm of the given number out of the table.*

Divide the logarithm thus found by the index of the root. Then the number answering to the quotient will be the root.

When the index of the logarithm to be divided is negative, and does not exactly contain the divisor without some remainder, increase the index by such a number as will make it exactly divisible by the index, carrying the units borrowed, as so many tens, to the left hand place of the decimal, and then divide as in whole numbers.

EXAMPLES.

1. What is the square root of 365?

Logarithm of 365 = 2.562293(2
Ans. 19.10469 = 1.281146$\frac{1}{2}$.

2. What is the third root of 12340?

Logarithm of 12340 = 4.091315(3
Ans. 23.108 = 1.363771$\frac{2}{3}$.

3. What is the seventh root of 6?

Logarithm of 6 = 0.778151(7
Ans. 1.2917 = 0.111164$\frac{3}{7}$.

4. Find the tenth root of 9.

Logarithm of 9 = 0.954243(10
Ans. 1.245 = 0.095424$\frac{3}{10}$.

5. Find the square root of .083.

Logarithm of .083 = —2.919078(2
Ans. .28809 = —1.459539.

6. Find the cube root of .00059.

Logarithm of .00059 = —4.770852(3
Ans. .083872 = —2.923617

Here the divisor 3, not being exactly contained in —4, it is augmented by 2, to make up 6, in which the divisor is contained just 2 times; then the 2 thus borrowed, being carried to the decimal figure 7, makes 27, which being divided by 3 gives 9, &c.

7. What is the value of x in the following equation?

$$x=\left(\frac{27\times38\times15.61}{.36\times1.37}\right)^{\frac{3}{4}}.$$

Log. 27 =1.431364
Log. 38 =1.579784
Log. 15.61 =1.193403
Log. .36=—1.556303 Arith. Com. =0.443697
Log. 1.37= 0.136721 " " =—1.863279

4.511527
3

13.534581(4

Ans. 2419.05=3.383645

8. Find the value of x in the following equation.

$$x=\frac{37}{223}\left(\frac{14.21\times.00208}{.035}\right)^{\frac{4}{3}}.$$

Ans. 2.853.

9. What is the value of x in the following equation?

$$x=\frac{7}{11}\left(\frac{144}{237}\right)^{\frac{2}{3}}\cdot\left(\frac{703}{819}\right)^{\frac{3}{5}}.$$

Ans. 3572.

10. Find the value of x in the following equation.

$$x=\frac{345}{417}\cdot\left(\frac{872\times.0065}{.038\times4685}\right)^{\frac{3}{5}}.$$

Ans.

11. What is the value of x in the following equation ?

$$\frac{25}{476} \cdot \left(\frac{873}{956}\right)^3 \cdot \left(\frac{278}{1973}\right)^{\frac{3}{4}}.$$

Ans.

SECTION XXVI.

COMPOUND INTEREST.

Art. 301. Compound Interest is interest charged not only on the principal, but also on the interest of preceding years.

Let p=principal,

r=rate per cent. considered as a decimal, or hundredths,

t=time in years,

A=amount.

Then $1+r$ will represent the amount of \$1, or £1, for one year.

And $p\ (1+r)$ will be the amount of any principal (p) for 1 year.

The amount for two years will be $p(1+r)\ .\ (1+r) = p(1+r)^2$; the amount for 3 years will be $p(1+r)^2\ .\ (1+r) = p(1+r)^3$; for 4 years it will be $p(1+r)^3\ .\ (1+r) = p(1+r)^4$.

Hence, for any number of years, it will be $p(+r)^n$; or $p(1+r)^t$.

Putting A for amount, we have the following formula for ascertaining the amount of any principal at any rate per cent. for any definite time at compound interest.

$$A=p(1+r)^t.$$

This equation contains four quantities, A, p, r, and t; any three of which being given, the other may be obtained.

Thus, we have the following

FORMULÆ.

1. $$A=p(1+r)^t.$$

2. $$p=\frac{A}{(1+r)^t}.$$

3. $$r=\left(\frac{A}{p}\right)^{\frac{1}{t}}-1.$$

4. $$t=\frac{\log.\left(\frac{A}{p}\right)}{\log.\ (1+r)}.$$

From the first formula, the pupil will perceive the following Rule may be deduced for finding the amount of any sum at compound interest.

RULE. *Add* 1 *to the ratio, then raise this sum to a power whose exponent is equal to the time, multiply this power by the principal, and the product is the amount.*

By logarithms the operation is much facilitated, especially when the time is of much length.

EXAMPLES.

1. What is the amount of $78.39 for 8 years, at 6 per cent. compound interest?

OPERATION BY THE FIRST FORMULA.

$$A=p(1+r)^t=78.39(1+.06)^t.$$

Log. $(1+r)=1.06$ $= 0.025306$
Multiply by $t=8$ 8
$(1+r)^t=(1.06)^8$ $= 0.202448$
Log. $p=78.39$ $= 1.893706$
$A=\$124.78$. *Ans.* $= 2.096154$

2. What is the amount of \$144, for 6 years, 9 months, at compound interest, at 5 per cent.?

Log. $1+r)=1.05$ $= 0.021189$

Multiply by t $\quad 6$

$(1+r)^t=(1.05)^t$ $= 0.127134$

Log. $p=144$ $= 2.158362$

Log. of amount for 6 years $= 2.285496$

Log. (1.0375) $= 0.015988$

$A=\$200.21$. *Ans.* $= 2.301484$

We have just found the logarithm of the amount for 6 years, and to this we have added the logarithm of 1.0375, it being the amount of \$1 for 9 months, at 5 per cent.

3. What is the compound interest of \$500 for 9 years, at 6 per cent. per annum, the interest to be paid semi-annually?

As the time t is to be calculated in half years, and as r is considered the interest of \$1 for one year, therefore $2t$ will represent the time, and $\frac{r}{2}$ the interest of \$1 for half a year. The formula will therefore be

$$A=p\left(1+\frac{r}{2}\right)^{2t}=500(1+.03)^{18}.$$

Log. $\left(1+\frac{r}{2}\right)=1.03$ $= 0.012837$

Multiply by 18 half years $\quad 18$

Log. $\left(1+\frac{r}{2}\right)^{2t}$ $= 0.231066$

Log. $p=500$ $= 2.698970$

A \$851.21. *Ans.* $\quad 2.930036$

4. What principal, at compound interest, will amount to $4000 in 10 years, at 6 per cent.?

This question must be performed by the second formula.

$$p=\frac{A}{(1+r)^t}=\frac{4000}{(1.06)^{10}}.$$

$$\begin{array}{lcl}
\text{Log. } 1.06=0.025306 & & \\
\qquad\qquad\quad \underline{10} & & \\
\qquad 0.253060 \text{ Arith. Com.} & = & -1.746940 \\
\text{Log. } A=4000 & = & \underline{3.602060} \\
\quad p=\$2233.57.\ \textit{Ans.} & = & 3.349000
\end{array}$$

5. At what rate per cent. must $2233.57 be at compound interest, to amount to $4000 in 10 years?

This question should be performed by the third formula.

$$r=\left(\frac{A}{p}\right)^{\frac{1}{t}}-1=\left(\frac{4000}{2233.57}\right)^{\frac{1}{10}}-1.$$

$$\begin{array}{lcl}
\text{Log. } A=4000 & = & 3.602060 \\
\text{Log. } p=2233.57 & = & \underline{3.349000} \\
 & & \underline{0.253060}(10 \\
\text{Log. } (1+r)1.06 & & 0.025306 \\
\qquad\qquad\ \ \underline{1} & & \\
\qquad\qquad .06,\ \text{that is, 6 per cent. } \textit{Ans.} & &
\end{array}$$

6. In what time will $2233.57, at compound interest at 6 per cent., amount to $4000?

This question is solved by the fourth formula.

$$t=\frac{\text{Log. }\left(\frac{A}{p}\right)}{\text{Log. }(1+r)}=\frac{\text{Log. }\left(\frac{4000}{2233.57}\right)}{\text{Log. }\ (1+.06)}=\frac{\text{Log.}4000-\log.2233.57}{\text{Log. }(1+.06)}$$

$$\begin{array}{lcl}
\text{Log. } A=4000 & = & 3.602060 \\
\text{Log. } p=2233.57 & = & \underline{3.349000} \\
 & & \underline{0.253060} \\
\text{Log. } (1+r)=1.06 & & 0.025306
\end{array}$$

Therefore $t=\frac{253060}{25306}=10$ years. *Ans.*

The value of this fraction can be ascertained by logarithms. Thus,

Log. 253060	= 5.403223
Log. 25306	= 4.403223
	1.000000

$t = 10$ years as before.

7. What will $16 amount to in 30 years, at 5 per cent. compound interest? *Ans.* $69.15.

8. What will $2000, at compound interest, amount to in 11 years, at 8 per cent.? *Ans.* $4663.34.

9. What will $27.18 amount to in 8 years, 3 months, at 4 per cent. compound interest? *Ans.* $37.56.

10. What is the compound interest of $1728 for 8 years, 6 months, at 6 per cent. per annum, the interest to be paid every 3 months? *Ans.* $3042.64.

11. What is the compound interest of $18.29 for 8 years, 8 months, 12 days, at 4 per cent.? *Ans.* $25.73.

12. What sum at compound interest will amount to $800 in 7 years, at 5 per cent. compound interest? *Ans.* $595.34.

13. What sum will amount to $500 in 9 years, at 6 per cent. per annum, the interest to be paid every 3 months? *Ans.* $292.54.5.

14. At what rate per cent. will $800, at compound interest, amount to $1609.76 in 12 years? *Ans.* 6 per cent.

15. In how many years will $3726 amount to $5007.43, at 3 per cent. compound interest? *Ans.* 10 years.

16. How many years will it require for any sum to double itself, at 6 per cent. compound interest?

Let $2p =$ the amount.

Then $2p = p(1+r)^t$.

And $2 = (1+r)^t$.

$$t = \frac{\text{Log. } 2}{\text{Log.}(1+r)}.$$

Log. 2 $= 0.301030$

Log. 1.06 $= 0.025306$

Therefore $\frac{301030}{25306} = 11.89$ years. *Ans.*

17. How many years will it require any sum to triple itself, at 5 per cent. compound interest?

Ans. 28 years, 151 days.

18. In 1840, the number of inhabitants in the United States was 17,068,666; in 1850, the number was 23,267,498. What was the gain per cent. per annum? *Ans.* .03146 per cent.

19. At the same rate as in the last question, in what year will there be 100,000,000 inhabitants? *Ans.* May 3d, 1897.

NOTE.—This answer is on the presumption, that the census is taken the first day of May.

20. Required the compound interest upon $155 for 9 years, at 3½ per cent. *Ans.* $56.24+.

21. Required the amount of $820 for 2½ years, at 4½ per cent. per annum, the interest being paid half-yearly.

Ans. $916.49+.

22. What sum at compound interest, for 2½ years, at 4½ per cent., the interest payable every six months, will amount to $458.25? *Ans.*

23. At what rate per cent. will $2000, at compound interest, amount to $4663.34 in 11 years? *Ans.*

SECTION XXVII.

DEPOSITS.

Art. 302. A deposit is a sum of money lodged in the hands of some person or corporation for safe keeping.

1. Deposited annually in a Savings Bank, which pays 6 per cent. compound interest, \$144 for 20 years. How much money shall I have in the bank at the end of the 20th year?

Let $a =$ the sum annually deposited,
$r =$ the rate of interest,
$t =$ the time,
$A =$ the amount.

By the rule of compound interest, the sum first deposited will amount to $144(1+.06)^{20}$, or $a(1+r)^t$; for the second year, $144(1+.06)^{19}$, or $a(1+r)^{t-1}$; for the third year, $144(1+.06)^{18}$, or $a(1+r)^{t-2}$; for the last year $144(1+.06)^1$, or $a(1+r)^1$.

303. We have now a regular series in Geometrical Progression, where the extremes are $a(1+r)^t$ and $a(1+r)^1$, the ratio $1+r$, to find the sum of the series.

Hence, by Art. 273 we have the following formula for obtaining the amount of the deposits.

$$A=\frac{a(1+r)[(1+r)^t-1]}{r}.$$

OPERATION BY LOGARITHMS.

Log. $(1+r)=1.06$		$=0.025306$
Multiply by $t=20$		$=\quad 20$
Log. $(1+r)^{t=20}$	3.207	$=0.506120$
Subtract	1	
Log.	2.207	$=0.343802$
Log. $(1+r)=1.06$		$=0.025306$
Log. $a=144$		$=2.158362$
Log. $r=.06=-2.778151$ Arith. Com.		$=1.221849$
Ans. \$5614.60		$=3.749319$

2. A gentleman has a daughter, who is 10 years old, and he wishes to give her, as soon as her age shall be 21 years, $2000. What sum must he deposit annually in a bank, which pays 5 per cent. compound interest, to be able to accomplish it?

304. The question given above may be solved by the following formula, which is obtained from the last by transposition, &c.

$$a=\frac{Ar}{(1+r)[(1+r)^{t}-1]}.=\frac{2000\times.05}{(1.05).[(1+.05)^{11}-1]}.$$

OPERATION BY LOGARITHMS.

Log. 2000		3.301030
Log. .05		—2.698970
	From	2.000000
Log. 1.05=0.021189		
11		
1.71=0.233079		
1		
Log. .71		=—1.851258
Log. 1.05		= 0.051189
	Take	—1.872447
Ans. $134.14.		= 2.127553

3. A gentleman, when his daughter was 10 years old, deposited for her annually $134.14 in a bank, which paid 5 per cent. compound interest. This sum remained until the time of her marriage. The amount then was $2000. What was then her age?

305. The formula for the operation of the above question is obtained from the former by transposition, &c.

$$t=\frac{\text{Log.}\left(\frac{Ar}{a(1+r)}+1\right.}{\text{Log.}\quad(1+r)}=\frac{\text{Log.}\left(\frac{2000\times.05}{134.14(1+.05)}\right)+1}{\text{Log.}\quad(1.05)}$$

Log. 2000 = 3.301030
.05 = —2.698970
From 2.000000
Log. 1.05 = 0.021189
Log. 134.14 = 2.127553
Take 2.148742
0.71 = —1.851258
1
1.71 = 0.232996

$.232996 \div .021189 = 11$ years nearly.

$10 + 11 = 21$ years. *Ans.*

4. A certain town in the United States, at the beginning of 1840 had 1000 inhabitants. There has been an immigration to this town each successive year, on the 1st of January, of 1000 additional inhabitants. Now supposing the population each year to gain 3 per cent., how many inhabitants would there be in this town at the end of 10 years? *Ans.* 11,809.

5. A gentleman, at the time of his marriage, deposited in a Savings Bank for the use of his wife, the sum of $150. This he continued to do for every six months until she was fifty years old. Now, if the bank pay a semi-annual dividend of 2 per cent. compound interest, and the gentleman's wife at the time of her marriage was 25 years old, what is the amount of the deposits? *Ans.* $12,939.97

6. If a man deposits annually in a bank $47, in how long time will it amount to $400, at 6 per cent. compound interest? *Ans.* 6 years, 273 days.

7. A gentleman has a son, who is 15 years old, and a daughter, who is ten years old. He intends that each of them, at the age of 21, shall have $5000 in a Savings Bank, which pays an annual dividend of 4½ per cent. What sum shall he deposit annually for each?

Ans. $712.33 for the son, $345.68 for the daughter.

8. Deposited annually in a bank which pays 4 per cent. compound interest, a certain sum, which in 10 years amounted to $300. What was the annual deposit? *Ans.* $24.02,6.

9. A certain young lady deposited $10 in a Savings Bank, and this she continued every three months. Now, if the bank pays 1½ per cent. compound interest at the end of each quarter, what will be the amount of her deposits in 10 years? *Ans.* $550.81.

10. Now, if the lady in the last question had deposited $40 annually at the commencement of the year, and had received 6 per cent. compound interest, would her deposits at the end of ten years have been more or less than before? *Ans.* $8.05 more.

SECTION XXVIII.

EXPONENTIAL OR TRANSCENDENTAL EQUATIONS.

Art. 306. To what power must 7 be raised to amount to 2401?

Let x be the power.

Then $7^x = 2401$.

The second power of 7 is found by multiplying the logarithm of 7 by 2; and the fifth power of 7 is found by multiplying the logarithm of 7 by 5; see Art. 300; therefore the xth power of 7 is found by multiplying the logarithm of 7 by x.

We have, therefore, the following equation, the logarithm of 7 being 0.845098, and the logarithm of $2401 = 3.380392$.

$$x \times 0.845098 = 3.380392.$$

Therefore, $$x = \frac{3.380392}{0.845098} = 4\text{th power. } \textit{Ans.}$$

The value of x is obtained by dividing the logarithm of the numerator by the logarithm of the denominator.

The value of the logarithms may also be obtained by subtracting the logarithm of the denominator from the logarithm of the numerator, and finding the value of the remainder. Thus,

Log. 3.380392 $=$ 0.528967
Log. 0.445098 $= -1.926907$
Ans. 4th power as before, $=$ 0.602060

307. If the form of the equation be $x^x=a$, the value of x may be found by the following

RULE. *First, find by trial two numbers as near the true value of* x *as possible, and substitute them for* x *separately. Then say, As the difference of the results is to the difference of the two assumed numbers, so is the difference of the true result and either of the former, to the difference of the true number and the supposed one belonging to the result last used. Add this difference to the supposed number, or subtract from it, according as it may be either too little or too great, and it will give the true value* nearly.

EXAMPLES.

1. What is the value of x in the following equation, $x^x=100$?

Here $x\times\log. x=\log. 100=2.$

We find the value of x upon trial to be between 3 and 4.

Log. 3= 0.477121
Log. 4= 0.602060
Log. 3×3= 0.477121×3 =1.431363
Log. 4×4= 0.602060×4 =2.408240
Difference of results =0.976883
2.000000
1.431363
Difference from the true result = .568637

Therefore, .976883 : 1 : : .568637 : .582

$3.+.582=3.582=x$ nearly.

This value of x is found on trial to be too small, and 3.6 is found to be too great, therefore, by substituting each of these we have

$$\begin{array}{lll}
\text{Log. } 3.582 & =0.554126 & \\
\text{Log. } 3.6 & =0.556303 & \\
\text{Log. } 3.582\times3.582 & =0.554126\times3.582 & =1.984879 \\
\text{Log. } 3.6 \quad \times36 & =0.556303\times3.6 & =\underline{2.002690} \\
 & & 0.017811
\end{array}$$

$3.6-3.582=.018$; $2.000000-1.984879=0.015121$.

Then $.017811 : .018 :: 0.015121 : .0152$,

Therefore $.0152+3.582=3.5972$ very nearly.

2. Given $x^x=10$ to find x.

First, let $x=2.5$,

Then log. $2.5= \qquad =0.397940$,

And $0.397940\times2.5= \qquad = .994850$;

Secondly, let $x=2.6$,

Then log. $2.6= \qquad =0.414973$,

And $0.414973\times2.6= \qquad =1.078929$,

$1.078929-.994850= \qquad = .084079$,

$1.-.994850=.005150$; $2.6-2.5=.1$,

Then $.084079 : .1 :: .005150 : .006$;

$2.5+.006=2.506$ nearly.

3. Required the value of x in the following equation.

$x^x=256$. *Ans.* $x=4$.

4. Given $x^x=5$ to find the value of x. *Ans.* $x=2.1249$.

5. Required the value of x in the following equation.

$7^x=2401$. *Ans.* $x=4$.

6. Find the value of x in the following equation, $x^x=3125$.

Ans. $x=5$.

308. This rule will apply to solving questions in geometrical progression, when we wish to obtain the number of terms.

EXAMPLES.

7. If the first term be 5, the last term 405, and the ratio 3, what is the number of terms?

In Art. 270, we find $L = ar^{n-1}$, and this equation by transposition, &c. is

$$n = \frac{\text{Log.}\left(\frac{L}{a}\right)}{\text{Log. } r} + 1 = \frac{\text{Log. } L - \text{Log. } a}{\text{Log. } r} + 1.$$

OPERATION.

Log. 405 = 2.607455
Log. 5 = 0.698970
1.908485

Log. 3=0.477121
Log. 1.908485 = 0.280688
Log. 0.477121 =—0.678628
Ans. 4 = .602060

4+1=5 the number of terms. *Ans.*

8. If the first term is 4, the ratio 3, and the sum of the series 484, what is the number of terms?

In Art. 274, we find

$$S = \frac{ar^n - a}{r-1} \text{ or } \frac{a(r^n-1)}{(r-1)}.$$

Therefore, by transposition, we have

$$\frac{\text{Log. } [a+(r-1)S] - \text{Log. } a}{\text{Log. } r} = \frac{\text{Log. } [4+(3-1)484] - \text{Log. } 4}{\text{Log. } 3}$$

$$[4+(3-1)484] - 4 = 972.$$

Log. 972 = 2.987666
Log. 4 = 0.602060
2.385606

Log. 3=.477121
Log. 2.385606 = 0.377598
Log. .477121 =—0.678628
= 0.698970

Ans. 5 the number of terms.

SECTION XXIX.

ANNUITIES.

ART. **309.** Annuity is a term used for any periodical income, arising from money lent, or from tenements, land, salaries, pensions, &c., payable from time to time, but generally by annual payments.

310. Annuities are divided into those that are in Possession, and those that are in Reversion; the former meaning such as have commenced, and the latter such as will not begin till some particular event has happened, or till after some certain time has elapsed.

311. When an annuity is forborne for some years, or the payment not made for that time, the annuity is said to be in *arrears.*

312. An annuity may also be for a certain number of years; or it may be without any limit, and then it is called a *perpetuity.*

313. The amount of an annuity, forborne for any number of years, is the sum arising from the addition of all the annuities for that number of years, together with the interest due upon each after it became due.

314. The *present worth* or value of an annuity, is the price or sum which ought to be given for it at the present time.

1. A man is desirous to bequeath his son a certain sum of money, which shall be deposited in an annuity office, that pays 6 per cent., that his son may receive at the close of each year $100 for the term of 12 years; at which time the prin-

cipal at interest should be exhausted. What is the sum bequeathed?

Let $A =$ the sum put at interest,
$a =$ the sum taken out annually,
$r =$ the rate per cent.,
$t =$ the time.

315. The amount of the sum, a, taken out at the close of the first year, would be at the end of the time $100(1+.06)^{11}$, or $a(1+r)^{t-1}$; that taken out at the close of the second year would amount to $100(1+.06)^{10}$, or $a(1+r)^{t-2}$; that taken out at the end of the third year would be $100(1+.06)^{9}$, or $a(1+r)^{t-3}$; that taken out at the end of the 12th year, would be only a, or $100 without interest.

Thus, we have a regular series in Geometrical Progression, where we have the extremes, a and $a(1+r)^{t-1}$, and the ratio $(1+r)$, given to find the sum of the series. Therefore by Art. 275 we find the sum of the terms to be $\frac{[a(1+r)^t-(1+r)]+a}{r}$. This sum must be equal to the amount of A for the given time. That is,

$$A(1+r)^t=a[(1+r)^t-(1+r)]+a.$$

Wherefore, $$A=\frac{a[(1+r)^t-(1+r)]}{r(1+r^t)}+\frac{a}{(1+r)^t}.$$

By reducing the terms, we have the following formula for obtaining the sum deposited.

$$A=\frac{a[(1+r)^t-1]}{r(1+r)^t}=\frac{100[(1.06)^{12}-1]}{.06(1+.06)^{12}}.$$

OPERATION BY LOGARITHMS.

Log. $1+r=1.06=0.025306$

12

$2.0122 \qquad =0.303672$

1

Log. $1.0122 \qquad =0.005266$

Log. $100 \qquad =2.000000$

From $\quad 2.005266$

Log. $(1+r)^t=(1.06)^{12} \qquad =0.303672$

Log. $r=.06 \qquad =-2.778151$

Take -1.081823

\$838.38. *Ans.* $=2.923443$

2. A gentleman deposited in an annuity office \$2000. How much can he receive annually, if the annuity continue 15 years at 5 per cent. compound interest?

By transposition, &c., of the last formula, we obtain the following for ascertaining the value of the annuity, a.

$$a=\frac{Ar(1+r)^t}{(1+r)^t-1}. \quad =\frac{2000\times.05(1.05)^{15}}{(1.05)^{15}-1}.$$

Log. $1+r=1.05= \qquad 0.021189$

15

$(1+r)^t=2.0789= \qquad 0.317835$

$1.$

Log. $1.0789=0.032981$ Arith. Com. $=-1.967019$

Log. $(A)=2000 \qquad = \; 3.301030$

Log. $(r)= \; .05 \qquad =-2.698970$

Log. $(1+r)^t=(1.05)^{15} \qquad = \; 0.317835$

$a=\$192.68.$ *Ans.* $= \; 2.284854$

In the operation of the above question, we find it more convenient to commence with the denominator of the formula.

3. A gentleman deposited in an annuity office, which pays 5 per cent. compound interest, $8000; in how many years will this sum be exhausted, if he draws out annually $850?

316. From the equation, $A=\frac{a[(1+r)^t-1]}{r(1+r)^t}$, we obtain by transposition, &c.,

$$t=\frac{\text{Log.}\left(\frac{a}{a-Ar}\right)}{\text{Log.}\ (1+r)}.=\frac{\text{Log.}\left(\frac{850}{850-(8000\times.05)}\right)}{\text{Log.}\ (1.05)}.$$

Log. $(A)=850$ $=2.929419$

$Ar=8000\times.05=400$

Log. $(850-400)=450$ $=2.653213$

0.276206

Log. $(1+r)=1.05$ $=0.021189$

Therefore $\frac{276206}{21189}=13.035=13$ years, 12 days. *Ans.*

317. But the same result will be obtained by subtracting the logarithm of the denominator from the logarithm of the numerator, and finding the number corresponding with the remainder. Thus,

Log. 276206 $=5.441233$

Log. 21189 $=4.326110$

Ans. $13.035=13$ years, 12 days, $=0.115123$

4. John Smith, believing he shall live 20 years, has purchased an annuity, which affords him $500 each year. What sum has he deposited in the annuity office, which pays for deposits 5 per cent. compound interest? The principal and interest are to be exhausted at the close of the 20th year.

Ans. $17,359.23.

5. If John Smith shall die at the end of 10 years, what sum will remain in the office? *Ans.* $8303.80.

6. Or, if the office have agreed, for his deposit, to give him, at the close of each year, \$500; and if Smith should live 30 years, what will the office lose?

Ans. The office will lose \$8303.80.

7. A gentleman bequeathed to his wife \$1728, which she deposited in an office which pays 4 per cent. compound interest. How large a sum shall she receive, annually, from the office, that the annuity may continue 10 years?

Ans. \$138.39.

8. A certain Savings Bank will pay $1\frac{1}{2}$ per cent. compound interest semi-annually. If I deposit in this bank \$4000, and take from it at the end of every six months \$500, in what time shall I have withdrawn all my money from the bank?

Ans. 3 years, 331 days.

9. What sum shall I deposit in an annuity office, that I may draw on it every 3 months for \$90? The bank pays on deposits 1 per cent. each quarter of the year, and I wish to continue drawing on the bank for 10 years.

Ans. \$4443.80.

SECTION XXX.

INVOLUTION OF BINOMIALS.

Art. 318. A binomial or residual quantity may be raised to any power without the trouble of continual involution, by the following

Rule. 1. *To find the terms without the coefficients.*

The index of the first, or leading quantity, begins with the index of the given power, and, in the succeeding terms, decreases continually by 1, *in every term to the last; and in the* *second, or following quantity, the indices of the terms are* 0, 1, 2, 3, 4, *&c. increasing by* 1. *That is, the first term will contain only the first part of the root with the same index as the required power. The last term of the series will contain only the second part of the given root, raised to the intended power; but all the other intermediate terms will contain the product of some powers of both members of the root, that the powers or indices of the first or leading member will always decrease by* 1, *while those of the second member will increase by* 1.

2. *To find the coefficients.*

The first coefficient is always 1, *and the second is the same as the index of the required power; to obtain the third coefficient, multiply that of the second term by the index of the leading letter in the same term, and divide the product by* 2; *and so on, that is, multiply the coefficient of the term last found by the index of the leading quantity in that term, and divide the product by the number of terms to that place, and it will give the coefficient of the term next following. In this manner, all the coefficients will be obtained.*

NOTE 1.—The whole number of terms will be one more than the index of the given power; and, when both terms of the root are +, all the terms of the power will be +; but, if the second term be —, all the odd terms will be +, and all the even terms —, which causes the terms to be + and — alternately.

NOTE 2.—The sum of the two indices in each term is always the same number; that is, the index of the required power; and reckoning from the middle of the series, both ways, or towards the right and left, the indices of the two terms are the same figures at equal distances, but mutually changed places. Also, the coefficients are the same numbers at equal distances from the middle of the series towards the right and left; so, by whatever numbers they increase to the middle, by the same, in the reverse order, they decrease to the end.

EXAMPLES.

1. Let $a+x$ be involved to the fifth power.

The terms without the coefficients by the first rule will be

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5.$$

The coefficients by the second rule will be

$$1, 5, \frac{5\times4}{2}, \frac{10\times3}{3}, \frac{10\times2}{4}, \frac{5\times1}{5}. =$$

$$1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

Therefore, the fifth power with the coefficients is

$$a^5,\ 5a^4x,\ 10a^3x^2,\ 10a^2x^3,\ 5ax^4,\ x^5.$$

2. Involve $a-x$ to the sixth power.

Ans. The terms with the coefficients will be,

$$a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6.$$

3. Required the 10th power of $a+x$.

Ans. $\begin{cases} a^{10}-10a^9x+45a^8x^2+120a^7x^3+210a^6x^4+252a^5x^5 \\ +210a^4x^6+120a^3x^7+45a^2x^8+10ax^9+x^{10}. \end{cases}$

4. Raise $x+y$ to the seventh power.

Ans. $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7.$

5. What is the 9th power of $a-b$?

Ans. $\begin{cases} a^9-9a^8b+36a^7b^2-84a^6b^3+126a^5b^4-126a^4b^5+84 \\ a^3b^6-36a^2b^7+9ab^8-b^9. \end{cases}$

The coefficients of the first twelve powers will be found in the following

TABLE.

First power,	1, 1
Second "	1, 2, 1
Third "	1, 3, 3, 1
Fourth "	1, 4, 6, 4, 1
Fifth "	1, 5, 10, 10, 5, 1
Sixth "	1, 6, 15, 20, 15, 6, 1
Seventh "	1, 7, 21, 35, 35, 21, 7, 1
Eighth "	1, 8, 28, 56, 70, 56, 28, 8, 1
Ninth "	1, 9, 36, 84, 126, 126, 84, 36, 9, 1,
Tenth "	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1
Eleventh "	1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1
Twelfth "	1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1.

By examining the above table, we readily perceive the law, by which the coefficients are obtained.

If we wished to obtain the coefficients of the 6th power, we add together the coefficients of the 5th power two and two.

Thus, $1+5=6$; $5+10=15$; $10+10=20$; $10+5=15$; $5+1=6$. By this process, we obtain all the coefficients of the 6th power, except the first and last, which are always 1 in every power.

To obtain the coefficients of the 10th power, we add those of the 9th. Thus,

$1+9=10$; $9+36=45$; $36+84=120$; $84+126=210$; $126+126=252$; $126+84=210$; $84+36=120$; $36+9=45$; $9+1=10$.

We therefore find the coefficients to be,

1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1.

6. Raise $a+4b$ to the third power.

Let $\quad x=4b.$

Then $\quad a+x=a+4b.$

The 3d power of $a+x$ by Art. 318, $=$

$$a^3+3a^2x+3ax^2+x^3.$$

Substituting $4b$ for x, we have

$$a^3+3a^2(4b)+3a(4b)^2+(4b)^3=$$
$$a^3+12a^2b+48ab^2+64b^3. \quad \textit{Ans.}$$

7. What is the 3d power of $a+b+c$?

Let $\quad n=b+c.$

Then $\quad a+n=a+b+c.$

The 3d power of $\quad a+n=a^3+3a^2n+3an^2+n^3.$

Substituting the values of x, we have

$$a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3=$$
$$\textit{Ans.}\begin{cases} a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2 \\ +b^3+3b^2c+3bc^2+c^3. \end{cases}$$

8. What is the 3d power of $a+b+c+d$?

Let $x=a+b$, and $y=c+d$.
Then $(x+y)^3=(a+b+c+d)^3$,
And $(x+y)^3=(x^3+3x^2y+3xy^2+y^3$.

Substituting these values of x and y, we have

$$(a+b)^3+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3=$$
$$a^3+3a^2b+3ab^2+b^3+(3a^2+6ab+3b^2)(c+d)+(3a+3b)(c^2+2cd+d^2)+(c^3+3c^2d+3cd^2+d^3)=$$
$$a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3a^2d+6abd+3b^2d+3ac^2+6acd+3ad^2+3bc^2+6cbd+3bd^2+c^3+3c^2d+3cd^2+d^3.$$ *Ans.*

9. What is the 3d power of $2a-b+c^2$? *Ans.*

10. What is the 5th power of $4a-5b$? *Ans.*

11. What is the 6th power of $3a^2-2b^3$? *Ans.*

12. What is the 4th power of $m+n-p$? *Ans.*

13. What is the 8th power of m^2+n^3? *Ans.*

14. What is the 7th power of $1+x^2$? *Ans.*

15. What is the 2d power of $a+b+c+d+e+f$? *Ans.*

16. What is the 10th power of a^3+b^3? *Ans.*

17. What is the nth power of $a+b$? *Ans.*

18. What is the 6th power of $a-b+c$? *Ans.*

19. What is the 4th power of a^5-x? *Ans.*

20. What is the 3d power of $2a^2-3b^3$? *Ans.*

21. What is the 3d power of $\sqrt{x}+\sqrt[3]{x^2}$? *Ans.*

22. What is the 2d power of $\sqrt{x^3}-\sqrt{y^2}+x^2$? *Ans.*

SECTION XXXI.

BINOMIAL THEOREM.

319. The Binomial Theorem is a general algebraical expression, or formula, by which any power, or root, of a given quantity, consisting of two terms, is expanded into a series; the form of which, as it was first proposed by Sir Isaac Newton, being as follows.

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}\left[1+\frac{m}{n}Q+\frac{m}{n}\left(\frac{m-n}{2n}\right)Q^2+\frac{m}{n}\left(\frac{m-n}{2n}\right)\right.$$
$$\left(\frac{m-2n}{3n}\right)Q^3+\frac{m}{n}\left(\frac{m-n}{2n}\right)\left(\frac{m-2n}{3n}\right)\left(\frac{m-3n}{4n}\right)Q^4+\text{\&c.}$$

Or,

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ$$
$$+\frac{m-3n}{4n}DQ+\text{\&c.},$$

where P is the first term of a binomial, Q the second divided by the first, $\frac{m}{n}$ the index of the power, or root, and A, B, C, &c. the terms immediately preceding those in which they are first found, including their signs $+$ or $-$.

320. This theorem may be applied to any particular case, by substituting the numbers, or letters, in the given example, for P, Q, m, and n, in either the above formulæ, and then finding the result according to the rule.

When the index of the binomial is a whole number, the series will terminate, as observed under the article Involution; but when it is a negative or fractional number, as in the following examples, the series will proceed on *ad infinitum*, and will become more convergent the less the second term of a binomial is with respect to the first.

EXAMPLES.

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Let $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$, and $n=2$.

Then $P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A$.

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B.$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x}{2.4a^3}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{x}{a^2}=\frac{3x^3}{2.4.6a^5}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=E.$$

$$\frac{m-4n}{5n}EQ=\frac{1-8}{10}\times-\frac{3.5x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=F.$$

Therefore $(a^2+x)^{\frac{1}{2}}=$

$$a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}-\&c.$$

The pupil will readily perceive, that the law of formation of the several terms of the series is sufficiently evident.

2. Required the development of $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}$ in a series.

Here $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}=x^2(x^2-y)^{-\frac{1}{2}}$, $P=x^2$, $Q=-\frac{y}{x^2}$, $m=-1$, and $n=2$.

Hence $P^{\frac{m}{n}}=(x^2)^{\frac{m}{n}}=(x^2)^{-\frac{1}{2}}=\frac{1}{x}=A$.

$$\frac{m}{n}AQ=-\tfrac{1}{2}\times\frac{1}{x}\times-\frac{y}{x^2}=\frac{y}{2x^3}=B.$$

$$\frac{m-n}{2n}BQ=\frac{-1-2}{4}\times\frac{y}{2x^3}\times-\frac{y}{x^2}=\frac{3y^2}{2.4x^5}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{-1-4}{6}\times\frac{3y^2}{2.4x^5}\times-\frac{y}{x^2}=\frac{3.5y^3}{2.4.6x^7}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{-1-6}{8}\times\frac{3.5y^3}{2.4.6x^7}\times-\frac{y}{x^2}=\frac{3.5.7y^4}{2.4.6.8x^9}=E.$$

Therefore $\frac{1}{(x^2-y)^{\frac{1}{2}}}=\frac{1}{x}+\frac{y}{2x^3}+\frac{3y^2}{2.4x^5}+\frac{3.5y^3}{2.4.6x^7}+\frac{3.5.7y^4}{2.4.6.8x^9}+\&c.$

And $\frac{x^2}{(x^2-y)^{\frac{1}{2}}}=x+\frac{y}{2x}+\frac{3y^2}{2.4x^3}+\frac{3.5y^3}{2.4.6x^5}+\frac{3.5.7y^4}{2.4.6.8x^7}+\&c.$

This last equation is obtained from the former by multiplying each term of the equation by x^2.

3. Required the cube root of 9.

Here, $9^{\frac{1}{3}}=(8+1)^{\frac{1}{3}}$,

Therefore, $P=8$, $Q=\frac{1}{8}$, $m=1$, and $n=3$.

Whence, $P^{\frac{m}{n}}=8^{\frac{m}{n}}=8^{\frac{1}{3}}=2=A.$

$$\frac{m}{n}AQ=\frac{1}{3}\times2\times\frac{1}{2^3}=\frac{1}{3.2^2}=B.$$

$$\frac{m-n}{2n}BQ=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=C.$$

$$\frac{m-2n}{3n}CQ=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=D.$$

$$\frac{m-3n}{4n}DQ=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.9.12.2^{10}}=E.$$

Therefore,

$$9^{\frac{1}{3}}=2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\text{ \&c.}$$

4. What is the square root of $a+b$?

Here, $\frac{m}{n}=\frac{1}{2}$, $P=a$, and $Q=\frac{b}{a}$.

Then, $P^{\frac{m}{n}}=a^{\frac{1}{2}}=A$,

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a^{\frac{1}{2}}}{1}\times\frac{b}{a}=\frac{a^{\frac{1}{2}}b}{2a}=\frac{a^{-\frac{1}{2}}b}{2}=B,$$

$$\frac{m-n}{2n}BQ=\frac{1-2}{4}\times\frac{a^{-\frac{1}{2}}b}{2}\times\frac{b}{a}=-\frac{a^{-\frac{1}{2}}b^2}{8a}=-\frac{a^{-\frac{3}{2}}b^2}{8}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{1-4}{6}\times-\frac{a^{-\frac{3}{2}}b}{8}\times\frac{b}{a}\times\frac{3a^{-\frac{3}{2}}b^3}{48a}=\frac{a^{-\frac{5}{2}}b^3}{16}=D.$$

And $\frac{m-3n}{4n}DQ=\frac{1-6}{8}\times\frac{a^{-\frac{5}{2}}b^3}{16}\times\frac{b}{a}=\frac{5a^{-\frac{5}{2}}b^4}{128a}=-\frac{5a^{-\frac{7}{2}}b^4}{128}=E.$

Therefore, $(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{a^{-\frac{1}{2}}b}{2}-\frac{a^{\frac{3}{2}}b^2}{8}+\frac{a^{\frac{5}{2}}b^3}{16}-\frac{5a^{-\frac{7}{2}}b^4}{128}$ &c.

5. What is the cube root of 7?

Ans. $2-\frac{1}{3.2_{\frac{1}{2}}}+\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}-$ &c.

6. Expand $(1-a)^{\frac{2}{5}}$ into an infinite series.

Ans. $1-\frac{2a}{5}-\frac{2.3.a^2}{5.10}-\frac{2.3.8.a^3}{5.10.15}-$ &c.

7. It is required to convert $\frac{1}{(1+x)^{\frac{1}{5}}}$, or its equal $(1+x)^{-\frac{1}{5}}$ into an infinite series.

Ans. $1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-$ &c.

8. It is required to convert $(a-b)^{\frac{1}{4}}$ into an infinite series.

Ans. $a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7.b^3}{4.8.12a^3}-\frac{3.7.b^4}{4.8.12.16a^4}-, \&c.\right)$

INDETERMINATE COEFFICIENTS.

321. This is a general method of obtaining a series from fractions and other expressions without either performing the division or extracting the root.

RULE. *Assume a series with unknown, but constant, coefficients of* x, *increasing or decreasing in the same way, as if the operation was performed at length; then make this series equal to the given expression, and, clearing the equation of fractions, bring all the terms to one side, so as to make the equation* $=0$; *next make the first term of the coefficients of the several powers of* x *each* $=0$, *and there will arise as many independent equations as there are unknown coefficients, from which their values may be found and substituted for them in the assumed series.*

EXAMPLES.

1. Let it be required to expand $\frac{a}{b+x}$ into a series.

Assume $\frac{a}{b+x}=A+Bx+Cx^2+Dx^3+$ &c.; then multiplying both sides by $b+x$, and transposing a, we obtain $Ab-a+(Bb+A)x+(Cb+B)x^2+(Db+C)x^3+\&c.=0$, an equation which must be true, whatever be the value of x. Now, making the first term and the coefficients of the several powers of x each $=0$, we have $Ab-a=0$, or $A=\frac{a}{b}$; $Bb+a=0$, or $B=\frac{A}{b}=-\frac{a}{b^2}$; $Cb+B=0$, or $C=\frac{B}{b}=+\frac{a}{b^3}$; $Db+c=0$, or $D=\frac{C}{b}=-\frac{a}{b^4}$, &c. And substituting these values of A,

B, C, D, &c. in the assumed series, we get $\frac{a}{b+x}=\frac{a}{b}-\frac{ax}{b^2}+\frac{ax^2}{b^3}-\frac{ax^3}{b^4}+$ &c., in which, it is obvious, that the signs are alternately + and —, and the exponents, both in the numerator and denominator, increase continually by 1, that of x in the numerator being always 1 less than that of b in the denominator.

2. Expand $\frac{a^2}{a^2+2ax-x^2}$ into a series.

Ans. $1-\frac{2x}{a}+\frac{5x^2}{a^2}-\frac{12x^3}{a^3}+$ &c.

3. Expand $\sqrt{(a^2-x^2)}$ into a series.

Ans. $a-\frac{x^2}{2a}-\frac{x^4}{8x^3}-\frac{x^6}{16a^5}-$ &c.

4. Expand $\frac{1+2x}{1-x-x^2}$ into a series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+$ &c.

This is a running series, in which each of the coefficients, after the second, is the sum of the two preceding ones.

5. Expand $\sqrt{(1-a)}$ into a series.

Ans. $1-\frac{a}{2}-\frac{a^2}{2.4}-\frac{3a^3}{2.4.6}-\frac{3.5a^4}{2.4.6.8}-\frac{3.5.7a^5}{2.4.6.8.10}-$ &c.

6. Expand $\frac{1-x}{1-2x-3x^2}$ into a series.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6$, &c.

7. What is the expansion of $(a-b)^{\frac{1}{4}}$?

Ans. $a^{\frac{1}{4}}\left(1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-,\ \&c.\right)$

SECTION XXXII.

SUMMATION AND INTERPOLATION OF SERIES.

Art. **322.** The Summation of Series is the method of finding a terminated expression equal to the whole series.

Interpolation is the method of finding any term of an infinite series without producing all the rest.

DIFFERENTIAL METHOD.

323. The Differential Method consists in finding, from the successive differences of the terms of a series, any intermediate term, or the sum of the whole series.

Problem I.

324. To find the several orders of differences.

Let $a+b+c+d+e+$ &c. be any series; subtract each term from the one following it, and the differences $-a+b$, $-b+c$, $-c+d$, $-d+e$, &c. will form a new series called the *first order of differences.* Again, subtract each term of this new series from the one that follows it, and the differences $a-2b+c$, $b-2c+d$, $c-2d+e$, &c. will form another series, called the *second order of differences.* Proceed in like manner for the third, fourth, fifth, &c. order of differences, until they at last become equal to 0, or are carried as far as is required.

325. When the several terms of the series continually increase, the differences will all be positive; but when they decrease, the differences will be alternately negative and positive.

1. Required the several order of differences of the series 1, 6, 20, 50, 105, 196, &c.

1, 6, 20, 50, 105, 196, &c. the given series.
5, 14, 30, 55, 91, &c. 1st differences.
9, 16, 25, 36, &c. 2d "
7, 9, 11, &c. 3d "
2, 2, &c. 4th "
0, &c. 5th "

2. Required the several order of differences of the series of 1^2, 2^2, 3^2, 4^2, 5^2, &c.

1, 4, 9, 16, 25, &c. the given series.
3, 5, 7, 9, &c. 1st differences.
2, 2, 2, &c. 2d "
0, 0, &c. 3d "

3. Required the several order of differences of the series of cubes, 1^3, 2^3, 3^3, 4^3, 5^3. *Ans.*

4. Find the order of differences in the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c. *Ans.*

PROBLEM II.

326. To find the first term of any order of differences.

Let d', d'', d''', d'''', &c. represent the first terms of the 1st, 2d, 3d, 4th, &c. order of differences; then $d'=-a+b$, $d''=a-2b+c$, $d'''=-a+3b-3c+d$, $d''''=a-4b+6c-4d+e$, &c.; from which it is obvious that the coefficients of the several terms of any order of differences are respectively the same as those of the terms of an expanded binomial, and are obtained in the same manner; for the terms that are subtracted are actually added, but with contrary signs. Hence, we infer, that d^n, or the first difference of the nth order of differences, is $\pm a\mp nb\pm n\,.\,\frac{n-1}{2}c\mp n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}d\pm$, &c. to $n+1$ terms; in which formula the *upper* signs must be taken when n is an even number, and the *under* when n is an odd number.

5. Required the first of the fifth order of differences of the series 6, 9, 17, 35, 63, 99, 148, &c.

Let a, b, c, d, e, f, &c. $=$ 6, 9, 17, 35, 63, 99, 148, &c., and $n=5$. Then

$$-a+nb-\frac{n(n-1)}{2}c+\frac{n(n-1)(n-2)}{2.3}d-\frac{n(n-1)(n-2)(n-3)}{2.3.4}e$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4}{2.3.4.5}f=-a+5b-\frac{5.4}{2}e+\frac{5.4.3}{2.3}d-$$

$$\frac{5.4.3.2}{2.3.4}e+\frac{5.4.3.2.1}{2.3.4.5}f=-6+45-170+350-315+99=$$

$494-491=+3$. *Ans.*

6. Required the first of the sixth order of differences of the series 3, 6, 11, 17, 24, 36, 50, 72, &c. *Ans.* —14.

PROBLEM III.

327. To find the *n*th term of the series a, b, c, d, e, f, &c.

As we have found in the last problem, that $d'=-a+b$, therefore $b=a+d'$, and, in the same manner, we find $c=a+2d'+d''$, $d=a+3d'+3d''+d'''$, $e=a+4d'+6d''+4d'''+d''''$, &c.; whence the *n*th term is

$$=a+\frac{n-1}{1}d'+\frac{n-1}{1}\cdot\frac{n-2}{2}d''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}d'''+\text{\&c.}$$

7. Required the 7th term of the series 3, 5, 8, 12, 17, &c.

3, 5, 8, 12, 17, &c. the given series.
2, 3, 4, 5, 1st difference.
1, 1, 1, 2d difference.
0, 0, 3d difference.

Here $d=2$; $d''=1$; $d'''=0$, and $n=7$.

Therefore $a+\frac{n-1}{1}d'+\frac{n-1}{1}\cdot\frac{n-2}{2}d''=3+\frac{7-1}{1}\cdot2+\frac{7-1}{1}\cdot\frac{7-2}{2}\cdot1=3+12+15=30=$ the 7th term.

8. Required the 9th term of the series 1, 5, 15, 35, 70, &c. *Ans.* 495.

9. Required the 10th term of the series 1, 3, 6, 10, 15, 21, &c. *Ans.* 55.

PROBLEM IV.

328. To find the sum of *n* terms of the series a, b, c, d, e, &c.

If we add the values of a, b, c, &c. as found in the last problem, we obtain $2a+d'=a+b$, $3a+3d'+d''=a+b+c$,

$4a+6d'+4d''+d'''=a+b+c+d$, &c. Wherefore, it is evident, that the sum of n terms must be

$$na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d''+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}d'''+,$$

&c.

329. When the differences become at last $=0$, any term or the sum of any numbers can be accurately found; but, when the differences do not vanish, the formulæ in this and the preceding problem give only an approximation, which will come nearer the truth, as the differences diminish.

10. Required the sum of 8 terms of the series 2, 5, 10, 17, &c.

Here $n=8$, $a=2$, $d'=3$, $d''=2$, and $d'''=0$.

Hence, $na+n.\frac{n-1}{2}d'+n.\frac{n-1}{2}.\frac{n-2}{3}d''=8.2+8.\frac{7}{2}.3+8.\frac{7}{2}.\frac{6}{3}.2=16+84+112=212=$ the sum of 8 terms.

11. Required the sum of 100 terms of the series, 1, 2, 3, 4, 5, &c.

Here 1, 2, 3, 4, 5, 6, &c. given series.
1, 1, 1, 1, 1, &c. 1st difference.
0, 0, 0, 0, &c. 2d difference.

Here $n=100$, $a=1$, and $d=1$

$$na+n.\frac{n-1}{2}d=100+100.\left(\frac{100-1}{2}\right)1=5050.\ \textit{Ans.}$$

12. Required the sum of 12 times of the series, 1, 4, 10, 20, 35. *Ans.* 1365.

13. Required the sum of n terms of the series, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, &c.

Here 1, 4, 9, 16, 25, 36, 49, &c. given series.
3, 5, 7, 9, 11, 13, &c. 1st difference.
2, 2, 2, 2, 2, &c. 2d difference.
0, 0, 0, 0, &c. 3d difference.

Let $a=1$, $d'=3$, and $d''=2$.

Then $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\,.\,3}=\frac{n(n+1).(2n+1)}{6}.$

14. Required the sum of n times of the series

$1^3, 2^3, 3^3, 4^3, 5^3, 6^3$, &c.; 1, 8, 27, 64, 125, 216, &c.

Here 1, 8, 27, 64, 125, 216, &c. given series.
7, 19, 37, 61, 91, &c. 1st difference.
12, 18, 24, 30, &c. 2d difference.
6, 6, 6, &c. 3d difference.
0, 0, &c. 4th difference.

Let $a=1$, $d'=7$, $d''=12$, $d'''=6$. Then

$$na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\,.\,3}d''+\frac{n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4}$$

$$d.'''=n+\frac{7n(n-1)}{2}+\frac{12n(n-1)(n-2)}{2\,.\,3}+\frac{6n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4.}=$$

$$n+\frac{7n^2-7n}{2}+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}=$$

$$\frac{4n}{4}+\frac{14n^2-14n}{4}+\frac{8n^3-24n^2+16n}{4}+\frac{n^4-6n^3+11n^2-6n}{4}=$$

$$\frac{n^4+2n^3+n^2}{4}=\frac{n^2(n+1)^2}{4}=$$ sum of n terms as required.

15. What is the number of cannon shot in a square pile, the bottom row consisting of 25 shot* ? *Ans.* 5525.

16. I have 10 square house lots, whose sides measure 5, 6, 7, 8, 9, &c. rods, respectively. What is their value at 5 cents per square foot? *Ans.* \$67,041.56¼.

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* Shot and shells are generally piled in three different forms, called triangular, square, or oblong piles, according as their base is either a triangle, a square, or a rectangle.

A square pile is formed by the continual laying of square, horizontal courses of shot, one above another, in such a manner, as that the sides of their courses decrease by unity from the bottom to the top row, which ends also in one shot.

17. There are 5 cubical blocks of marble, whose sides measure, respectively, 2, 3, 4, 5, and 6 feet? What is their value at $2.75 per cubic feet? *Ans.* 3368.75.

18. What is the number of shot in a square pyramidical pile, whose side at the base contains 100 shot? *Ans.*

19. What is the sum of 20 terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, 6^3, &c.? *Ans.*

20. What is the sum of 20 terms of the series 1^4, 2^4, 3^4, 4^4, 5^4, 6^4, &c.? *Ans.*

330. To find a fraction, that will express the value of a geometrical series to infinity.

In Art. 280, we find, that the sum of an infinite series is obtained by the following formula:

$$S=\frac{a}{1-r}.$$

and, by this formula, we may find the sum of algebraic series.

EXAMPLES.

1. What is the sum of the series $1+a+a^2+a^3+a^4$, &c. carried to infinity? *Ans.* $\frac{1}{1-a}$.

By the above formula, the first term of the series will be the numerator of the fraction, and the denominator is obtained by subtracting the second term from the first.

2. What fraction will express the exact value of the series $1+5+25+125$, &c. to infinity? *Ans.* $\frac{1}{1-5}$.

3. What fraction will express the infinite series $1-a+a^2-a^3+a^4-a^5$, &c.? *Ans.* $\frac{1}{1+a}$.

4. What fraction will express the series $\frac{h}{a}+\frac{bh}{a^2}+\frac{b^2h}{a^3}+$, &c. to infinity? *Ans.* $\frac{h}{a-b}$.

5. What is the sum of the series $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+$, &c. to infinity? *Ans.* $\frac{x-1}{x^2}$.

6. What fraction will express the series $1+2+3+4+5+$, &c. to infinity? *Ans.*

7. What fraction is equal to the series $\frac{1}{a}-\frac{x^2}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7}+$, &c. to infinity? *Ans.*

8. What fraction will express the value of $1+1+1+1$, &c. to infinity? *Ans.*

9. Express by a fraction the value of the series $x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+$, &c. to infinity? *Ans.*

10. What is the value of the series $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+$, &c. to infinity? *Ans.*

DISCOUNT AND PRESENT VALUE AT COMPOUND INTEREST.

Art. 331.

Let $p =$ the present value.
$s =$ the sum due.
$t =$ the time.
$d =$ the discount.

Then, by principles before explained, we have the following

FORMULÆ.

$$(1.)\qquad p=\frac{S}{(1+r)^t}.$$

$$(2.)\qquad d=\left(1-\frac{1}{(1+r)^t}\right).$$

EXAMPLES.

1. What is the present worth of $600, due 3 years hence, at 6 per cent. compound interest? *Ans.* $503.77.

2. John Smith, Jr. owes me $312.50, which is due 2 years hence, at 4½ per cent. compound interest. What sum will now discharge the debt? *Ans.* $286.55.

3. What is the present value of $1000, due 4 years hence, at 5 per cent. compound interest? *Ans.* $824.34.

4. What is the discount on $3700, due 10 years hence, at 5 per cent. compound interest? *Ans.* $1428.52.

5. What is the present worth of $3456, due 5 years hence, at 6 per cent. compound interest? *Ans.* $2582.52.

6. What is the discount on $1000, due four years hence, at 6 per cent. compound interest? *Ans.* $207.91.

7. Rented a house for 5 years, at $400 a year, the rent to be paid quarterly. What is the present worth of this rent, at 8 per cent. compound interest? *Ans.*

8. Loaned a friend $100 for one year, at 2 per cent. per month, compound interest; that is, the interest is to be added to the principal each month. What is the amount at the close of the year, and what is its present value? *Ans.*

9. Which is the greater present value, $400 due three years hence, at 5 per cent. compound interest, or $500 due 4 years hence, at simple interest? *Ans.*

10. What sum shall I put into the Savings Bank, which pays 5 per cent. compound interest, that shall in 6 years amount to $1000? *Ans.*

MISCELLANEOUS QUESTIONS.

1. Find the sum of $9x$ and $-17x$. *Ans.*

2. Find the sum of a and $-h$. *Ans.*

3. Find the sum of $-7a$ and $-11a$. *Ans.*

4. Find the sum of $2\sqrt{ax}-ax$, $-\sqrt{ax}$, $2xy$, and $-\sqrt{x}$. *Ans.*

5. From $-11x$ take $-21x$. *Ans.*

6. From $-7a$ take $9a$. *Ans.*

7. Reduce $2a+b$, $-(3a-5b)$, $+(8a-3b)-(a-b)$ to its simplest form. *Ans.*

8. Find the continued product of $a-a-a+a$. *Ans.*

9. Multiply $7\sqrt{x}+3\sqrt{x^3}-\sqrt{x^n}$ by $0\sqrt{y^3}$? *Ans.*

10. Multiply a^2+b^2 by $a^{-2}-b^{-2}$. *Ans.*

11. Multiply a^m+b^n by $a^{-2m}+b^{-n}$. *Ans.*

12. Multiply $\sqrt{ax}$ by $-\sqrt{ax}$. *Ans.*

13. Divide $-a$ by $-3a$. *Ans.*

14. Divide a^{n-m} by a^n. *Ans.*

15. Divide a^5+x^5 by $a+x$. *Ans.*

16. Multiply y^m+x^m by $y-x$. *Ans.*

17. Multiply $\frac{1}{xn}-\frac{na}{x^{n+1}}+\frac{n^2a^2}{2x^{n+2}}$ by $x^n+nax^{n-1}+\frac{n^2a^2x^{n-2}}{2}$. *Ans.*

18. Divide $1-n^8$ by $1-x$. *Ans.*

19. Multiply $3\sqrt[3]{x-a^2}$ by $4\sqrt{x^2-a}$. *Ans.*

20. Multiply $2\sqrt{x^2-x}$ by $4\sqrt{x^2+x}$. *Ans.*

21. Given $(x-1)\sqrt{2x-x^2}=\frac{1}{2}$ to find x.

Ans. $x=\frac{\sqrt{x}\pm 1}{\sqrt{2}}$.

22. Given $x-2\sqrt{x+2}=1+\sqrt[4]{x^3-3x+2}$ to find x.

Ans. $x=9\pm 4\sqrt{7}$.

23. Given $\frac{x}{2}-\frac{5x+4}{3}=\frac{4x-9}{3}$ to find the value of x.

Ans. $x=\frac{2}{3}$.

24. Given $\frac{x-7}{11}-\frac{3x-5}{7}+\frac{125}{77}=2x-17$ to find the value of x.

Ans. $x=8$.

25. Given $\frac{5x-7}{3}-\frac{3x-2}{7}=\frac{x-5}{4}$ to find the value of x.

Ans. $x=\frac{67}{83}$.

26. Given $\frac{x+3}{2}-\frac{11-x}{5}=\frac{3x-1}{20}+3\frac{1}{5}$ to find the value of x.

Ans. $x=7$.

27. Given $\frac{\frac{3}{4}x-2}{8}+19=x-\frac{\frac{1}{2}x+1-\frac{1}{5}(x-4)+x}{11}$ to find the value of x.

Ans. $x=24$.

28. Given $\frac{17-3x}{5}-\frac{2x+1}{1\frac{1}{2}}=\frac{29-11x}{3}$ to find the value of x.

Ans. $x=4$.

29. Given $\frac{41-35x}{105}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{2x-2\frac{1}{5}}{6}$ to find the value of x.

Ans. $x=4$.

30. Given $\sqrt{x+9}=1+\sqrt{x}$ to find the value of x.

Ans. $x=16$.

31. Given $\frac{3-2x}{1-2x}-\frac{5-2x}{7-2x}=1-\frac{4x^2-2}{7-16x+4x^2}$ to find the value of x.

Ans. $x=\frac{7}{8}$.

32. Given $(\sqrt{x}+28)(\sqrt{x}+6)=(\sqrt{x}+38)(\sqrt{x}+4)$ to find the value of x.

Ans. $x=4$.

33. Given $\sqrt{4+\sqrt{x^4-x^2}}=x-2$ to find the value of x.

Ans. $x=2\frac{1}{8}$.

34. Given $\sqrt[3]{ax+b}=\sqrt[3]{cx+d}$ to find the value of x.

Ans. $x=\frac{d-b}{a+c}$.

35. Given $\sqrt[m]{a+x}=\sqrt[2m]{x^2-5ax+b^2}$ to find the value of x.

Ans. $x=\frac{a^2-b^2}{3a}$.

36. Given $+b^2=a^2+bx$ to find the value of x.

Ans. $x=a+b$.

37. Given $\frac{5}{6}(x-a)-\frac{1}{5}(2x-3b)=10a+11b$ to find the value of x.

38. Given $\frac{3ac}{a+b}+\frac{a^2b}{(a+b)^2}+\frac{(2a+b)bx}{a(a+b)^3}=\frac{3cx}{b}+\frac{x}{a}$ to find the value of x.

Ans. $x=\frac{ab}{a+b}$.

39. Given $\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}=b^{\frac{1}{2}}$ to find the value of x.

Ans. $x=\frac{2ab^{\frac{1}{2}}}{1+b}$.

40. Given $\frac{(a+x^{\frac{1}{2}})^{\frac{1}{2}}}{x^{\frac{1}{4}}}+\frac{(a-x^{\frac{1}{2}})^{\frac{1}{2}}}{x^{\frac{1}{4}}}=x^{\frac{1}{4}}$ to find the value of x.

Ans. $x=4(a-1)$.

41. Given $\left(\frac{a^2-x}{x}\right)^{\frac{1}{2}}-\left(\frac{a^2}{x}-b\right)^{\frac{1}{2}}=c^{\frac{1}{2}}$ to find the value of x.

Ans. $\frac{4a^2c}{4b^2+c^2}$.

42. A gentleman travelled 252 miles. The first day he rode 4 miles, the last 128, and each day's journey was double the preceding one. How many days was he performing the journey?

Ans.

43. A gentleman dying left his sons an estate of $13,187.50. He bequeathed to his youngest son $1000 ; to the oldest $5062.50 ; and ordered that each son's portion should exceed the next younger by the ratio of $1\frac{1}{2}$. How many sons had he ?
Ans. 5 sons.

44. The first term is 3, the last term $\frac{1}{9}$, and the sum of the series $4\frac{4}{9}$. What is the number of terms ? *Ans.* 4.

45. The first term is $\frac{1}{5}$, the ratio 7, and the last term $3361\frac{2}{7}$. What is the number of terms ? *Ans.* 6.

46. What are the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$?
Ans. $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

47. Required the sum of 200 terms of the series 1, 3, 5, 7, 9, &c. *Ans.*

48. The first term of an arithmetical series is —7, the tenth term is 12. What is the sum of the series ? *Ans.*

49. If a man travel 20 miles the first day, and 15 miles the second, and so continue to travel 5 miles less each day, how far will he have advanced on his journey the 8th day ?
Ans.

50. The first term of an arithmetical series is 5, the number of terms 20; what must the common difference be, that the sum of the series shall be $123\frac{1}{2}$? *Ans.*

51. If a man travel 20 miles the first day, 19 the second day, $18\frac{1}{20}$ the third day, and so on in a geometrical progression, in how many days will he have travelled 400 miles ?
Ans.

52. A merchant, having mixed a certain number of gallons of wine and water, found, that if he had mixed 6 gallons more of each, there would have been 7 gallons of wine to every 6 gallons of water ; but, if he had mixed 6 gallons less of each,

there would have been 6 gallons of wine to every 5 gallons of water. How much of each did he mix?

Ans. 78 gallons of wine with 66 of water.

53. A person bought 2 cubical stacks of hay for £41; each of them cost as many shillings per solid yard, as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each? *Ans.* £25 and £16.

54. A certain man owes $1000. What sum shall he pay daily, so as to cancel the debt, principal and interest, at the end of the year, reckoning it at 6 per cent. simple interest?

Ans.

55. A and B travelled on the same road, and at the same rate, from Portland to Boston. When A was at 50 miles distance from Boston, he overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and, two hours afterwards, met a stage wagon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese when he was at 45 miles distance from Boston, and met the stage wagon exactly forty minutes before he came at 31 miles distance from Boston. Where was B when A arrived at Boston? *Ans.* 25 miles from Boston.

56. A gentleman has two sons, John and Nathan. John is 10 years old, and Nathan is 15. He wishes to divide $1000 between his sons, in such a manner, that each, by depositing his share in a Savings Bank, which pays 5 per cent. compound interest, shall have the same amount in the bank when he is 21 years old. What sum shall each deposit? *Ans.*

57. My garden is 100 feet square, and I wish to raise its surface two feet with the soil taken from a ditch with which I intend to surround it. This ditch is to be 5 feet deep, and it is required to find its width. *Ans.* —9.1+ feet.

58. A engaged to reap a field for $10, which he would do in ten days; but after he had labored 2 days he engaged B, as he appeared to be a *smart* hand, to assist him, and he supposed he could finish the field in 3 days. But B proving to be a very inefficient workman, he was obliged to hire C, who proved to be a superior laborer, and the field was completed in 5 days. Now, if he had not hired C, and A and B had completed the work themselves, B would have received 1.08\frac{8}{19}$ in addition to his services for his 3 days' labor. How long would it have required B and C, each, to reap the field?

Ans. { B could have reaped the field in 11$\frac{1}{9}$ days,
C in 8$\frac{16}{23}$ days.

59. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he would have been 6 hours longer in performing the same journey. How many miles did he go per hour? *Ans.* 7 miles.

60. The difference between the hypothenuse and base of a right-angled triangle, is 6 feet; and the difference between the hypothenuse and perpendicular, is 3 feet. What are the sides of the triangle? *Ans.* 15, 12, and 9 feet.

61. In a parcel, which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and the whole is worth 18 shillings. How many are there of each? *Ans.* 6 of one, and 18 of the other.

62. The income of a certain estate is to be sold for a term of 7 years. A offers to pay $300 dollars *down*, and $300 at the end of each year; B offers $800 *down*, and $250 at the close of each year; C says he will pay $1300 *down*, and $200 annually; D will pay $2500 "*cash down.*"

Which has made the best offer, if interest is to be reckoned at 6 per cent. compound interest?

Ans. { Value of A's offer, $1974.71.4;
B's offer, $2195.59.5; C's offer, 2416.47.6;
D's offer, 2500. Hence D's offer is the best.

63. A gentleman being asked the age of his two sons, replied, that if the sum of their ages was multiplied by the age of the oldest, the product would be 144; but if the difference of their ages was multiplied by the younger, the product would be 14. What was the age of each? *Ans.*

64. The sum of two numbers is 20, and the sum of their cubes is 2060. What are the numbers? *Ans.*

65. If the product of two numbers be added to the square of the larger, the sum would be 112; but, if their product be taken from the square of the least, the remainder would be 24. Required the numbers. *Ans.*

66. What number is that, which, being added to twice its square root, the sum shall be 24? *Ans.*

67. If a man owe $2000, what sum shall he pay daily, so as to cancel the debt, principal and interest, at the end of the year, reckoning the interest at 6 per cent.? *Ans.* $5.6395.

68. I have $84\frac{1}{2}$ square feet of plank, that is 3 inches thick. How large a cubical box can be made from it?

Ans. Each side measures 48 inches.

69. From $62\frac{5}{72}$ feet of plank, that is $2\frac{1}{2}$ inches thick, I wish to make a box whose length is four times its width, and its height and width are to be equal. What are its dimensions?

Ans. Length 8 feet, width and height 2 feet.

70. There was a cask containing 20 gallons of wine; a certain quantity of this was drawn off into another cask of equal size, and this last filled with water, and afterwards the first cask was filled with the mixture. It now appears that, if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of wine in each. What was the quantity of wine drawn off at first?

Ans. 10 gallons.

71. After A had travelled at the rate of 4 miles an hour, had been set out two hours and three quarters, B set out to

overtake him, and in order thereto went four miles and a half the first hour, four and three quarters the second, five the third; and so on, gaining a quarter of a mile every hour. In how many hours would he overtake A? *Ans.* 8 hours.

72. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers. *Ans.* 5, 10, 20, 40.

73. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Ans. 14, 10, 6, 2.

74. A tailor bought a piece of cloth for £147, from which he cut off 12 yards for his own use, and sold the remainder for £120 5*s*., gaining 5 shillings per yard. How many yards were there, and what did it cost him per yard?

Ans. 49 yards, at £3 per yard.

75. In a mixture of rye and wheat, the difference between the quantities of each is to the quantity of wheat as 100 is to the number of bushels of rye; and the same difference is to the quantity of rye as 4 to the number of bushels of wheat. How many bushels are there of each?

Ans. 25 bushels of rye, and 5 of wheat.

76. It is required to find two numbers, such, that the product of the greater, and the square root of the less, may be equal to 48; and the product of the less, and the square root of the greater, may be 36. *Ans.* 16 and 9.

77. What two numbers are those, whose difference being multiplied by the greater, and the product divided by the less, is 48; but if their difference be multiplied by the less, and the product divided by the greater, the quotient is 3?

Ans. 16 and 4.

78 Find two numbers, such that the square of the greater, multiplied by the less, may be equal to 448; and the square of the less multiplied by the greater, may be 392.

Ans. 8 and 7.

79. Required those two numbers, which, being both multiplied by 27, the first product is a square, and the second the root of that square; but being both multiplied by 3, the first product is a cube, and the second the root of that cube.
Ans. 243 and 3.

80. A farmer has two cubical stacks of hay. The side of one is 3 yards longer than the side of the other, and the difference of their contents is 117 solid yards. Required the side of each. *Ans.* 5 and 2 yards.

81. There are two square buildings, that are paved with stones a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately? *Ans.* 26 and 38 feet.

82. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?
Ans. 14, 10, 6, 2.

83. A gentleman asking a lady her age, she replied, If you add the square root of it to half of it, and subtract 12, there will remain nothing. Required her age. *Ans.* 16.

84. What number is that, to which if 1, 7 and 19 be added, the first shall have the same ratio to the second, that the second has to the third? *Ans.* 5.

85. What are those two numbers, which have the same ratio to each other, that their difference has to 40?
Ans. 2 and 10.

86. There are two numbers, whose product is 54, and the greater of which is to the less as their sum is to 10. What are those numbers? *Ans.* 9 and 6.

87. Divide 20 into two such parts, that the square of the greater shall be to the square of the less as 9 to 4. What are those parts? *Ans.* 12 and 8.

88. Let 24 be divided into two such parts, that the quotient of the greater divided by the less, shall be to the quotient of the less divided by the greater, as 9 to 1? *Ans.* 18 and 6.

89. Divide 14 into two such parts, that their squares shall be to each other as 9 to 16. *Ans.* 6 and 8.

90. Divide 12 into two such parts, that the sum of their squares shall be to the difference of their squares as 5 to 3. *Ans.* 4 and 8.

91. Let 15 be divided into two such parts, that the square of the first shall be to the square root of the second as 12 to 1. *Ans.* 6 and 9.

92. There are two numbers, whose product is 12, and the sum of their cubes is to the cube of their sum as 91 to 303 *Ans.* 3 and 4.

93. What two numbers are those, whose difference, sum, and product, are as the numbers 3 and 5?

94. What two numbers are those, whose product is 120; and if the greater be increased by 8 and the less by 5, the product of the two numbers shall be 306? *Ans.* 10 and 12.

95. A, B, and C make a joint stock. A puts in $60 less than B, and $68 more than C; and the sum of the shares of A and B is to the sum of the shares of B and C as 5 to 4. What did each put in? *Ans.* A put in $140, B $200, and C $72.

96. A and B engage in speculation with different sums: A gains $150, B loses $50. Now A's stock is to B's as 3 to 2; but had A lost $50, and B gained $100, then A's stock would have been to B's as 5 to 9. What was the stock of each? *Ans.* A's $300, B's $350.

97. Find two numbers in the proportion of 5 to 7, to which two other required numbers, in the proportion of 3 to 5, being

respectively added, the sum shall be in the proportion of 9 to 13, and their difference shall be 16.

Ans. { First two numbers 30 and 42; last two numbers, 6 and 10.

98. A merchant mixes wheat, which costs 10 shillings per bushel, with barley which costs him 4 shillings per bushel, in such proportion, as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11 shillings per bushel. Required the proportion.

Ans. He must mix 14 bushels of wheat with 9 of barley.

The following questions may be performed when the pupil shall have read either the NATIONAL ARITHMETIC, or some treatise on Geometry.

99. There is a loaf of bread, which is half a sphere, whose base measures 12 inches. Required how thick the crust must be baked, that the remainder shall be half the contents of the loaf. *Ans.* .8038+ inch.

100. There are two towers of unequal heights, situated on a plane near each other. A line extending from the base of the less to the top of the larger is 100 feet; and a line from the base of the larger to the top of the less is 80.27+ feet; a perpendicular let fall from the point where the lines cross each other to the surface of the plane is 32 feet. Required the height of the towers and their distance from each other.

Ans. { Height of the larger tower, 80 feet; height of the less, $53\frac{1}{2}$ feet; distance between the towers, 60 feet.

101. There is a conical glass, 6 inches deep, the diameter at the top is five inches, and it is $\frac{1}{5}$ full of water. If a ball 4 inches in diameter be put into this glass, it is required to find how much of its axis is immersed in the water. *Ans.* .546 inch.

102. It is required to find how many balls one inch in diameter may be put into a cubical box, whose sides measure each one foot in the *clear*. *Ans.* 2151 balls.

A

TABLE,

CONTAINING THE

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

Numbers from 1 to 100 and their Logarithms, with their Indices.

No.	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

NOTE.—In the following part of the Table the Indices are omitted, as they can be very easily supplied by the directions given in Section xxv., p. 254, on Logarithms.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	010300	010724	011147	011570	011993	012415	424
3	012837	013259	013680	014100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	020361	020775	416
5	021189	021603	022016	022428	022841	023252	023664	024075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	030195	030600	031004	031408	031812	032216	032619	033021	404
8	033424	033826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426	7825	8223	8620	9017	9414	9811	040207	040602	040998	397
110	041393	041787	042182	042576	042969	043362	043755	044148	044540	044932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	050380	050766	051153	051538	051924	052309	052694	386
3	053078	053463	053846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	060320	379
5	060698	061075	061452	061829	062206	062582	062958	063333	063709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	070038	070407	070776	071145	071514	370
8	071882	072250	072617	072985	073352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	079543	079904	080266	080626	080987	081347	081707	082067	082426	360
1	082785	083144	083503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	090258	090611	090963	091315	091667	092018	092370	092721	093071	352
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	100026	346
6	100371	100715	101059	101403	101747	102091	102434	102777	103119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	110253	338
9	110590	110926	111263	111599	111934	112270	112605	112940	113275	3609	335
130	113943	114277	114611	114944	115278	115611	115943	116276	116608	116940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	120245	330
2	120574	120903	121231	121560	121888	122216	122544	122871	123198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	130012	323
5	130334	130655	130977	131298	131619	131939	132260	132580	132900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	140194	140508	140822	141136	141450	141763	142076	142389	142702	314
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	146438	146748	147058	147367	147676	147985	148294	148603	148911	309
1	9219	9527	9835	150142	150449	150756	151063	151370	151676	151982	307
2	152288	152594	152900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	160168	160469	160769	161068	301
5	161368	161667	161967	162266	162564	162863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	170262	170555	170848	171141	171434	171726	172019	172311	172603	172895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	176381	176670	176959	177248	177536	177825	178113	178401	178689	289
1	8977	9264	9552	9839	180126	180413	180699	180986	181272	181558	287
2	181844	182129	182415	182700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	190051	281
5	190332	190612	190892	191171	191451	191730	192010	192289	192567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	200029	200303	200577	200850	201124	274
9	201397	201670	201943	202216	202488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	204391	204663	204934	205204	205475	205746	206016	206286	206556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	210051	210319	210586	210853	211121	211388	211654	211921	267
3	212188	212454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	220370	220631	220892	221153	221414	221675	221936	222196	222456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	230193	256
170	230449	230704	230960	231215	231470	231724	231979	232234	232488	232742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	240050	240300	250
4	240549	240799	241048	241297	241546	241795	242044	242293	2541	2790	249
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	250176	245
8	250420	250664	250908	251151	251395	251638	251881	252125	252368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	255514	255755	255996	256237	256477	256718	256958	257198	257439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	260071	260310	260548	260787	261025	261263	261501	261739	261976	262214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	270213	270446	270679	270912	271144	271377	271609	233
7	271842	272074	272306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	278982	279211	279439	279667	279895	280123	280351	280578	280806	228
1	281033	281261	281488	281715	281942	282169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
5	290035	290257	290480	290702	290925	291147	291369	291591	291813	292034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	302331	302547	302764	302980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
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5	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
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7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
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3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958	370143	370328	370513	370698	370883	185
5	371068	371253	371437	371622	371806	1991	2175	2360	2544	2728	184
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6	390935	391112	391288	391464	391641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
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6	471292	471438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
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5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
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6	9093	9198	9302	9406	9511	9615	9719	9824	9928	620032	104
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8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
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2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	630021	630123	630224	630326	102
7	630428	630530	630631	630733	630835	630936	1038	1139	1241	1342	102
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
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3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
6	9335	9432	9530	9627	9724	9821	9919	650016	650113	650210	97
7	650308	650405	650502	650599	650696	650793	650890	0987	1084	1181	97
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
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2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
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4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
7	9916	660011	660106	660201	660296	660391	660486	660581	660676	660771	95
8	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
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6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
7	9317	9410	9503	9596	9689	9782	9875	9967	670060	670153	93
8	670246	670339	670431	670524	670617	670710	670802	670895	0988	1080	93
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
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2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	680063	680154	680245	91
9	680336	680426	680517	680607	680698	680789	680879	0970	1060	1151	91
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4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
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6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
9	9309	9398	9486	9575	9664	9753	9841	9930	690019	690107	89
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2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
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4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
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5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
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7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
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9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
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5	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
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2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	720077	83
5	720159	720242	720325	720407	720490	720573	720655	720738	720821	0903	83
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
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1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
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4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
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6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	730055	730136	730217	730298	730378	730459	730540	730621	730702	81
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
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5	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
9	9572	9651	9731	9810	9889	9968	740047	740126	740205	740284	79
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1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
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1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
2	9736	9814	9891	9968	750045	750123	750200	750277	750354	750431	77
3	750508	750586	750663	750740	0817	0894	0971	1048	1125	1202	77
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
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2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
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4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
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2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
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4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
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3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	778151	778224	778296	778368	778441	778513	778585	778658	778730	778802	72
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3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
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2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
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4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
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2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
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2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
4	8886	8953	9021	9088	9156	9223	9290	9358	0425	9492	67
5	9560	9627	9694	9762	9829	9896	9964	810031	810098	810165	67
6	810233	810300	810367	810434	810501	810569	810636	0703	0770	0837	67
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
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2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
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4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
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9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
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6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
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7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
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6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
5	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	770042	74
9	770115	770189	770263	770336	770410	770484	770557	770631	770705	0778	74
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1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
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2	9596	9669	9741	9813	9885	9957	780029	780101	780173	780245	72
3	780317	780389	780461	780533	780605	780677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	785401	785472	785543	785615	785686	785757	785828	785899	785970	71
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
6	9581	9651	9722	9792	9863	9933	790004	790074	790144	790215	70
7	790285	790356	790426	790496	790567	790637	0707	0778	0848	0918	70
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
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6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
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3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
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4	8886	8953	9021	9088	9156	9223	9290	9358	0425	9492	67
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8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
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650	812913	812980	813047	813114	813181	813247	813314	813381	813448	813514	67
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3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
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5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
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7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
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9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
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5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
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9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
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5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
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5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
8	850033	850095	850156	850217	850279	850340	850401	850462	850524	850585	61
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
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3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
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1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
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5	860338	860398	860458	860518	860578	0637	0697	0757	0817	0877	60
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
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1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
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1	9818	9877	9935	9994	870053	870111	870170	870228	870287	870345	59
2	870404	870462	870521	870579	0638	0696	0755	0813	0872	0930	58
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	875061	875119	875177	875235	875293	875351	875409	875466	875524	875582	58
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
8	9669	9726	9784	9841	9898	9956	880013	880070	880127	880185	57
9	880242	880299	880356	880413	880471	880528	0585	0642	0699	0756	57
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2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
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4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
5	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
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1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	890030	890086	890141	890197	890253	890309	890365	56
7	890421	890477	890533	0589	0645	0700	0756	0812	0868	0924	56
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
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4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
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2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
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6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
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2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
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1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	910037	53
3	910091	910144	910197	910251	910304	910358	910411	910464	910518	0571	53
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
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1	9601	9653	9706	9758	9810	9862	9914	9967	920019	920071	52
2	920123	920176	920228	920280	920332	920384	920436	920489	0541	0593	52
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
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3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
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3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
5	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
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9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
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4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
5	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
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9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
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7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
8	3445	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
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2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
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4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
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1	9878	9926	9975	950024	950073	950121	950170	950219	950267	950316	49
2	950365	950414	950462	0511	0560	0608	0657	0706	0754	0803	49
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
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3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
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1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
2	9995	960042	960090	960138	960185	960233	960281	960328	960376	960423	48
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
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2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	968530	968576	968623	968670	968716	968763	968810	968856	968903	47
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
3	9882	9928	9975	970021	970068	970114	970161	970207	970254	970300	47
4	970347	970393	970440	0486	0533	0579	0626	0672	0719	0765	46
5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
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1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
5	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	977769	977815	977861	977906	977952	977998	978043	978089	978135	46
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
5	980003	980049	980094	980140	980185	980231	980276	980322	980367	980412	45
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	982316	982362	982407	982452	982497	982543	982588	982633	982678	45
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	986817	986861	986906	986951	986996	987040	987085	987130	987175	45
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
7	9895	9939	9983	990028	990072	990117	990161	990206	990250	990294	44
8	990339	990383	990428	0472	0516	0561	0605	0650	0694	0738	44
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	991270	991315	991359	991403	991448	991492	991536	991580	991625	44
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	995679	995723	995767	995811	995854	995898	995942	995986	996030	44
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
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